Limit and Shakedown Analysis of Structures under Stochastic Conditions

Von der
Fakultät Architektur, Bauingenieurwesen und Umweltwissenschaften
der Technischen Universität Carolo-Wilhelmina
zu Braunschweig

zur Erlangung des Grades eines
Doktor-Ingenieurs (Dr.-Ing.)
genehmigte

Dissertation
von
Ngọc Trình Trần

geboren am 31.10.1981
aus Hà Tĩnh, Vietnam

Eingereicht am: 11. Juli 2018
Disputation am: 11. September 2018

Berichterstatter: Prof. Hermann G. Matthies, Ph.D.
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For my Mom and my Dad

Abstract

The plastic collapse limit and the shakedown limit are important limit states in modern ultimate limit state design. Due to the high expenses of experimental setups and the time consuming full elastic-plastic cyclic loading analysis, the determination of these limits by means of numerically direct plasticity methods has been of great interest to many designers. Lower bound limit analysis determines directly the largest load, which is safe against plastic collapse, as a maximum problem formulated in static quantities. Alternatively upper bound limit analysis determines the least collapse load as a minimum problem formulated in kinematic quantities. Both optimization problems are convex so that by duality they have the same solution which is therefore an exact solution of classical plasticity. Shakedown analysis extends the optimization approach to time variant loading and is used for limit state design to check against failure by alternating plasticity and incremental plastic collapse (ratcheting). A structure is safe against plastic failure if initial plastic deformations cease because the structure "shakes down" to elastic behavior.

Limit and shakedown analysis solve the plasticity problems by mathematical programming. If the characteristics of structures such as strength and loads are considered as random variables, shakedown analysis can be stated as a stochastic programming problem. The thesis contributes an approach to show that direct structural reliability design can be achieved on the basis of the required failure probabilities by chance constrained programming, which is an effective approach in stochastic programming. In the general case this is a hard problem because probabilities have to be calculated as high dimensional integrals during the optimization algorithm. The thesis developed successfully three algorithms to treat large-scale shakedown analysis problems with random strength and load variables. For random loads or the case of random strength and loads a kinematic algorithm (algorithm A1) has been developed. It allows to compute limit and shakedown loads for deterministic strength and loads; random strength with normal or lognormal distribution and normally distributed loads. Algorithm A2 has been developed to calculate lower bound and upper bound shakedown loads simultaneously in case of random strength random and deterministic loads acting on the structure. Algorithm A3 is a dual algorithm permitting the computation of limit and shakedown loads of a Kirchhoff-Love plate under uncertain conditions of strength. It calculates simultaneously upper and lower bound shakedown loads for normally or for lognormally distributed strength.

The First Order Reliability Method (FORM) is used for an independent check of the chance constrained programming solutions. If the deterministic problem has an analytical solution and strength and load are both either normally or lognormally distributed, then the so-called reliability index can also be computed analytically and the failure probability obtained. The latter is the starting point of the probabilistic limit state design proposed in this dissertation.

Keywords: Probabilitstic limit state design, Limit and shakedown analysis, Direct methods, Chance constrained programming, Kirchhoff-Love plate, ES-FEM.

Zusammenfassung

Die plastische Kollaps- und die Einspielgrenze sind wichtige Grenzzustände beim modernen Tragfähigkeitsnachweis. Aufgrund des hohen Aufwands von Versuchsaufbauten und der zeitaufwendigen vollständigen elastisch-plastischen zyklischen Belastungsanalyse ist die Bestimmung dieser Grenzen mittels numerischer direkter Plastizitätsmethoden für viele Konstrukteure von großem Interesse. Die untere Schrankenmethode der Traglastanalyse ermittelt direkt die größte Belastung, die gegen plastischen Kollaps sicher ist, als in statischen Größen formuliertes Maximumproblem. Alternativ dazu ermittelt die obere Schrankenmethode der Traglastanalyse die geringste Kollapslast als Minimumproblem in kinematischen Größen. Beide Optimierungsprobleme sind konvex, so dass sie wegen Dualität die gleiche Lösung haben, die also eine exakte Lösung der klassischen Plastizität ist. Die Einspielanalyse erweitert den Optimierungsansatz auf zeitlich veränderliche Belastungen und wird für den Grenzzustandsentwurf zur Überprüfung gegen Versagen durch alternierende Plastizität und inkrementellen plastischen Kollaps (Ratchetting) verwendet. Eine Struktur ist sicher gegen plastisches Versagen, wenn die anfänglichen plastischen Verformungen aufhören, weil sich die Struktur auf ein rein elastisches Verhalten einspielt.

Traglast- und Einspielanalyse lösen Plastizitätsprobleme durch mathematische Programmierung. Wenn die Eigenschaften von Strukturen wie Festigkeit und Belastung als Zufallsvariablen betrachtet werden, kann die Einspielanalyse als stochastisches Programmierproblem formuliert werden. Die Dissertation entwickelt einen Ansatz, mit dem eine direkte probabilistische Bemessung auf der Grundlage der erforderlichen Versagenswahrscheinlichkeiten erreicht werden kann durch wahrscheinlichkeitsrestringierte Programmierung, die ein effektiver Ansatz in der stochastischen Programmierung ist. Im allgemeinen ist dies ein schwieriges Problem, da Wahrscheinlichkeiten als hochdimensionale Integrale während des Optimierungsalgorithmus berechnet werden müssen. Die Dissertation entwickelte erfolgreich drei Algorithmen zur Behandlung von großen Einspielanalyse-Problemen mit zufälligen Variablen für Festigkeit und Belastung: Für zufällige Lasten oder für zufällige Festigkeiten und Lasten wurde ein kinematischer Algorithmus (Algorithmus A1) entwickelt. Er erlaubt die Berechnung von Traglast und Einspiellasten für deterministische Festigkeiten und Belastungen; zufällige Festigkeiten mit Normal- oder Lognormalverteilung und normalverteilte Belastungen. Algorithmus A2 wurde entwickelt, um bei zufälligen und deterministischen Belastungen, die auf die Struktur einwirken, gleichzeitig die untere und obere Grenze der Einspiellast zu berechnen. Algorithmus A3 ist ein dualer Algorithmus, der die Berechnung von Traglast und Einspiellasten einer Kirchhoff-Love-Platte unter ungewissen Festigkeitsbedingungen ermöglicht. Er berechnet gleichzeitig obere und untere Lastgrenzen für normal oder für logarithmisch verteilte Festigkeiten.

Die First Order Reliability Method (FORM) wird für eine unabhängige Überprüfung der Lösung der wahrscheinlichkeitsrestringierten Programmierung verwendet. Hat das deterministische Problem eine analytische Lösung und sind Festigkeit und Belastung beide entweder normal oder lognormal verteilt, so kann auch der sogenannte Zuverlässigkeitsindex analytisch berechnet und die Versagenswahrscheinlichkeit ermittelt werden. Letztere ist Startpunkt des in dieser Dissertation vorgeschlagenen probabilistischen Grenzzustands-entwurfs.

Stichworte: Probabilistischer Grenzzustandsentwurf, Traglast- und Einspielanalyse, direkte Methoden, wahrscheinlickeitsbeschränkte Programmierung, Kirchhoff-Love-Platte, ES-FEM.

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ACKNOWLEDGEMENTS

In 2012 I started as a PhD student at the Biomechanics Laboratory, FH Aachen University of Applied Sciences, Campus Jülich. My journey to Germany arrived to its end, with all its beauties and became to a part of my life. The journey would not have succeeded without the cooperation and encouragement I received all along.

My thanks go to the Vietnamese government and Hanoi Architectural University via program 322 that opened the door of research for me in Germany. I also want to thank FH Aachen University of Applied Sciences, which also supported me financially and with good conditions for my research activities.

My deepest and most sincere gratitude goes to my supervisors Professor Dr. Hermann G. Matthies , Professor Dr. Georgios E. Stavroulakis and Professor Dr.-Ing. Manfred Staat who have guided me through the doctoral research. I would like to thank them for giving me support and encouragement. Special thanks to you for your kindness and enthusiasm, for providing me valuable suggestions in research. Without your support, my thesis would not have been possible. Additionaly, I want to thank all of committee members - Professor Dr. sc. techn. Klaus Thiele, Professor Dr.-Ing. Dieter Dinkler - for taking their time to read and comment my thesis.

I wish to express many thanks to my former colleague Dr.-Ing. Trần Thanh Ngọc for his help and useful disscusions.

I wish to deeply thank all the members of the Laboratory of Biomechanics of FH-Aachen, the Institute for Scientific Computing Tecnische Universität Braunschweig for their support.

I wish to express many thanks to Professor Hà Huy Cương, Professor Nguyễn Xuân Ngọc, Dr. Trịnh Tự Lực, Dipl.-Ing. Karl-Heinz Gatzweiler for their strong encouragement.

Very special thanks go to my mother Nguyễn Thị Thanh Hiên, my father Trần Văn Chương, my brother Trần Ngọc Anh and my sister Trần Ngọc Thu Hoài for their love, unconditional support and encouragement.

Trần Ngọc Trình

Jülich, September 2018

Chapter 1

Introduction

1.1 Limit and Shakedown Analysis of Structures

Direct methods comprising limit and shakedown analysis is a branch of computational mechanics. It plays a significant role in mechanical and civil engineering design. The concept of direct method aims to determinate the ultimate load bearing capacity of structures beyond the elastic range.

Theory of limit analysis allows to estimate the maximum load intensity that structures can support under proportional and gradually increasing loading. The limit analysis methods are based on the theorems of plastic failure of bodies made from elastic-perfectly plastic or rigid-plastic material. These theorems are known as lower bound (static) and upper bond (kinematic) theorem of limit analysis. There are two approaches corresponding to these theorems, static approach and kinematic approach. Accordingly, the static theorem leads to solving a maximum mathematical optimization problem while the kinematic theorem leads to solving a minimum optimization problem. In comparison to incremental analysis (step-by-step method), the efficiency this 'direct method' is achieved by observing only the ultimate state, the failure state, irrespective of the detailed information about what happened to the structure.

Despite the fact that some ideas appeared in 18th century, the limit analysis is of a later date. Its beginnings are associated with Kazincy (1914), who calculated the failure load of a beam clamped at both ends and confirmed this result by experiments. A similar concept was proposed in 1917 by Kist and in 1926 by Grüning. However, the early works in this area relied mostly on the intuitional point of view of engineers. Although the first static theorem was first proposed by Kist (1917) as an axiom, it is considered that basic theorems of limit analysis wer first laid out by Gvozdev in 1936. The first papers considering the limit analysis of some simple structures appeared more than eighty year ago [188]. Apart from the possibility to permit higher loads, limit analysis seemed to be easier to use than the classical elastic structural analysis. Theoretical foundations of the whole approach were developed slightly later in some works, for instant in work of Drucker, Greenberg and Prager [186].

Accompanied by the development of the finite element method and the rapid evolution of computer technology in the 1960s, 1970s, the direct method appears more and more such as works of Biron and Hodge [1], Hodge and Belytschko [2], Neal [3], Maier [4], Casciaro[5], Morelle [6], Drosopoulos and Stavroulakis [7] and many more. Limit analysis based on mathematical programming, the solution were not always easy to obtain. Therefore, researches were executed to find efficient algorithms for linear and nonlinear programming. Linear programming has been used widely in limit analysis because this approach permits the solving of large scale problems. Contributions for this approach can be seen in [8], [9], [10], [11],.[12]. Following the investigation of Overton [13] that the limit analysis problem can be solved efficiently using a Newton-type scheme, many works on the development of new algorithms have been carried out using anonlinear yield function mostly von Mises. Contributions in this direction are the work of Gaudrat [14], Liu *et al*. [15], [16], Capsoni and Corradi [17] , Christiansen *et al*. [18].

The theory of limit analysis based on the assumption that the loads acting on a structure are monotonically increasing. However, in practice, loads are often functions of time and may vary independently. Moreover, they may be cyclic loads repeated many times. In this situation structures can fail even if the loads remain inside the elastoplastic domain of the load space. Clearly, a more general concept of safety is necessary to cover these situations. Under general, non-monotonic loading, the structure can fail due to incremental plastic collapse (or ratcheting) and low cycle fatigue (or alternating plasticity). On the contrary, if after some time plastic strains cease to develop and the structure responds purely elastically to the variable loads, the structure is said to shakedown. Under the assumptions of the existence of a convex yield surface and the validity of the normality rule for the plastic strain rates, Melan 1938 [19] and Koiter 1960 [20] gave upper and lower bound criteria, respectively, for shakedown or non-shakedown. However, the classical shakedown theory of Melan and Koiter are limited to elastic-perfectly plastic material behavior.

The extension to bounded kinematic hardening has been achieved in theory and solved with different direct numerical methods [174-178]. The analytical solution of the cyclic tension-torsion test with a two surface plasticity model shows that kinematic hardening has only a restricted influence on shakedown loads [179-180]. The extension of shakedown analysis to damage and the inclusion of geometric effects is discussed in [181].

The problem of shakedown analysis can be solved using ‘’step-by-step’’ procedures in which the loading path is divided into small loading increments with a full analysis of the evolution of stresses and strains. The disadvantage of these methods is that many load cycles need to be calculated and the loading history has to be given, which is not realistic in many technical situations. Direct methods for the computation of shakedown loads overcome this issue, an advantage of direct method is that the loading history is not necessary to be known exactly, but only its bounds [21], [22], [23], [24], [25], [189].

Most often shakedown problems were discretized using the mesh-based methods such as finite element method, smoothed finite element method [26], boundary element method [3–5]. Recently, meshless methods have received much attention such as the element free Galerkin (EFG) method, the meshless local Petrov-Galerkin (MLPL) method, the reproducing kernel particle method (RKPM) [30], [31]. Among researchers who used meshless method to formulate limit and shakedown problems are Liu *et al.* [32], Vaghefi *et al.* [33], Xia *et al.* [34], Le *et al.* [35], Ho *et al.* [36]. The shakedown problems also are formulated using iso-geometric analysis such as [37], [38].

For practical problems, the direct methods lead to nonlinear convex optimization problems with a large number of variables and constraints. Therefore, many investigations were carried out to find effective algorithms for the solution of shakedown problem based on upper bound and lower bound shakedown theorems. The direct methods are based on mathematical programming. FEM discretizations of limit and shakedown analysis of realistic structural problems typically lead to 10^4 to 10^6 constraints. Therefore, the European research project LISA has been initiated by M. Staat and P.D. Panagiotopoulos with the objective to develop optimization methods for large-scale problems [22], [172-173]. The different developed numerical methods of the LISA project are a basis reduction lower bound method [182], [190], [192] the solution by second order cone programming [183], and a primal dual method [184], which is also used in this thesis. Based on Melan's lower bound shakedown, Simon and Weichert [13–15] investigated interior-point algorithm to solve the problem for the computation of shakedown loads of engineering structures subjected to varying thermo-mechanical loading. Based on the kinematic theorem, Tran *et al.* [42] develop a kinematic algorithm to treat the problem shakedown analysis of shells. Andersen *et al*. [47] developed an excellent primal-dual interior-point algorithm for minimizing a sum of Euclidean norms. Their works showed that applying of properties of duality associated with a Newton method may lead to very accurate results in limit analysis. Based on Andersen's study, Vu *et al.* [48], developed an dual algorithm to calculate shakedown loads of structures in which the upper bound shakedown load and lower bound shakedown load can be computed simultaneously. Algorithms based on second order cone programming have been used for direct methods in recent year such as [37], [49]–[52].

It is assumed in the conventional shakedown analysis that such characteristics of the structures as strength of material, dimension and shape of structures, loads have certain deterministic values. In fact, however, there is a certain degree of uncertainty associated with these parameters which may have non-negligible effects on the safety of the structure. As the result of the necessity to account in a rational way for such uncertainties, the theory of structural reliability has been introduced and has developed rapidly since 1970s up to now to provide a conceptually and operationally satisfactory design methodology [53], [54].

Present structural reliability analysis is typically based on the limit state of initial or local failure. This may be defined by first yield or by some member failure if the structure can be designed on an element basis. However, this gives quite discouraging reliability estimates for ductile materials because virtually all structures are redundant. Progressive member failures of such systems reduce redundancy until finally the statically determinate system fails. This system approach is not defined in an obvious way for a finite element representation of a structure. Moreover the progressive failure is a time variant problem. Probabilistic limit and shakedown analyses is a more effective method of structural reliability analysis. It is based on the direct computation of the load-carrying capacity or the safety margin. Most important, this approach makes the problem time invariant. Probabilistic limit and shakedown analyses were first proposed by Augusti, Baratta, Casciati [59]. Further work seemed to remain restricted to stochastic limit analysis of frames based on linear programming [53]. In the work of Siemaszko *et al*. [60], the shakedown and limit reliability problem was formulated in the load space and solved by simulation methods and by first order reliability methods (FORM). Staat and Heitzer

[61], [62], [63] extended plastic reliability analysis towards non-linear programming and used the lower bound theorems of limit and shakedown load to define a limit state function for reliability analysis by FORM. They also showed that seond order reliability methods (SORM) are not important for limit and shakedown analysis because the limit state function is typically linear or only slightly nonlinear. A simple linear search algorithm was used to calculate the design point. Following this direction, Tran *et al.* [43] presented a new kinematic algorithm of probabilistic limit and shakedown analysis for thin plates and shells. The advantage of the method in this direction is that sensitivity analyses are obtained directly from a mathematical optimization with no extra computation cost.

In the above mentioned works, the problems of structual reliability have been carried out through two stages: the first stage is the computation of limit (shakedown) load factors, at the second stage the failure probability is calculated by analytical or different numerical approaches. *One question which can be posed in engineering design is how to compute load factors if the reliability level or its failure probability of structures is prescribed*. This problem is more necessary in practical design and this issue was first investigated by Sikorski and Borkowski [64]. They showed that in limit analysis it is possible to evaluate the safety factor taking directly into account the random scatter of loading and material data using a method called chance constrained programming. This approach is based on the theory of stochastic programming and on the reliability analysis of complex systems. However, they were restricted to some simple structures such as two span continuous beams, trusses and simple frames. Further work was done by Tin-Loi [65], [66]. Tin-Loi *et al*. generalized the stochastic ultimate load analysis models used in structural engineering and proposed methods for solving them. Accordingly, the approach provided a direct way of assessing the probabilistic collapse safety. On the other hand, there are no applications to demonstrate the proposed method or to extend it to shakedown. Important other restrictions of the work of Sikorski and Borkowski and Tin-Loi et al. are only normal distribution, linear yield function (Tresca), only limit analysis and simple frame structures.

Inspired by the work of Sikorski and Borkowski, the aim of this thesis is to solve large-scale stochastic shakedown problems with a chance constrained programming approach.

1.2 Chance-Constrained Programming

Uncertainties are a built-in character of nearly all practical processes, in engineering problems , in finance problems, etc.. They have a considerable impact on the process in many situations. Therefore, optimization of such processes must be considered under these uncertainties. There are many approaches proposed to carry out this concept. The constributions include approximate polyhedral dynamic programming [69], [70], [71], nomial solutions [72], measurement-based optimization [73], [74], worst-case and distributional robustness analysis [75], [76], [77], robust optimization [78], recourse programming [79]–[81] and chance constrained optimization (CCOPT) [82], [83] . Robust optimization and recourse programming are two alternative approaches to optimization under uncertainty, which can be applied in settings slightly different from that of CCOPT. In this thesis, the CCOPT approach is used to treat the problem of shakedown analysis of structures under conditions of uncertainty.

Chance-constrained programming were first proposed in 1958 by Charnes, Cooper and Symonds [83], [82]. After that, chance-constrained programming has been applied in many areas such as water reservoir management [84], optimal power flow (OPF) [85], financial risk management using risk metrics like value at risk (VaR) and conditional value at risk (SVaR) [86], [87] , reliability based engineering design optimization (RBDO) [88]–[90], Control and optimization based on prediction, optimal and reliable wind (solar) power generation [78]and many more..

The main idea of CCOPT is to require the satisfaction of process restrictions with a predescribed probability level. An advantage of CCOPT is that a relationship between the optimality and the reliability can be obtained. Based on this relationship a compromised decision which balances the profitability and reliability can be made.

CCOPT problems can be classified considering several aspects. There are four different aspects are considered: type of model equations, linearity, temporal behavior of the uncertain vaialbes and distribution of the uncertain variables. *Type of model equations* can be described by an algebraic equation if the process model is static .They can be described by differential or difference equations if they are dynamic. *Linearity* of CCOPT can be seen at functions containing uncertain variables that can depend linearly or non-linearly on the uncertain input variables. Linear constraints are of special interest in connection with Gaussian distributed uncertainties. From these aspect, there are 16 different types of CCOPT problem can be found from all possible combinations of the single components. Each of these 16 different CCOPT problems has unique properties which allow certain techniques to be used in the solution process.

In general, a CCOPT solver is based on a standard NLP solver. In order to be able to use such solvers, the chance constraints have to be converted into equivalent deterministic constraints. CCOPT is a hard problem, the major difficulty is the calculation probabilistic constraints. There are several approaches to the evaluation of chance constraints such as the *Direct computation method, Linearization methods, Projection approaches and Analytical Approximation*. The direct computation approach is the first one used to solve CCOPT problems, it was proposed by Charnes et al., who introduced this approach together with CCOPT [82]. Even today it still fiunds widespread application, due to its convenience of usage and low demand on computational power. The major disadvantage of this approach is that it only works for (multivariate) Gaussian distribution of uncertainties and only for linear inequatity of random parameters.

Linearization methods, the second approach, are based on the idea of the extention of the direct computation method to nonlinear problems by using a first order Taylor series expansion. However, care should be takenwhen using such techniques in the presence of larger variances of the uncertain variables because these might lead to considerable errors in the gradients computation and, therefore, to problems in the solution of the optimization problem as shown in the work of Garnier et al.[91]. Wendt et al in[92] proposed the projection methods which cover a wide range of nonlinear problems. Futher more, these methods are not limited to a certain kind of uncertainty. The idea of these method is to look for a monotonic relationship between one of the uncertain input variables and their function. This relation can be used to convert the chance constraint into the domain of the uncertain input variables, where the probabilities can be computed using multivariate integration.

The approaches of analytical approximation was pioneered by Pinter [93] and generalized by Nemirovski and Shapiro [94]. These approaches use ideas from approximation of probability inequalities. The advantages of this approach are: the results in a smooth approximation of chance constraints, iterate of the approximation problem are always feasible to the chance constrained problem, it can be used irrespective of the distribution of the random variable. However, it still has some disadvantage such as there can be a large discrepancy between the chance constraint and its approximation. Very little work has been done in this direction. Recently, Getelu [95] introduced a new analytical approximation approach which can be easily generalized to a number of single chance-constraints.The advantages of this method is that the approximation does not depend on the distribution function of random variables, it is expected to provide a closer approximation of single chance-constraints. Approximate solutions remain always feasible to the (CCOPT). The difficulty of this approach is that the objective and constraints may require the evaluation of high dimensional integrals and the convexity structures of CCOPT may not be preserved in the new approximation. Currently, the approximation method works only for single chance constraints.

The approaches, which have been discussed above, convert chance constraints into equivalent deterministic constraints. They are the most important methods for transforming the probabilistic constraints into deterministic ones. They typically require faster evaluation of multidimensional probability integrals for large-scale problems. This can be achieve through efficient numerical integration techniques full grids, sparse grids, quasi-Monte-Carlo cubature.

Full grid methods are also called full-grid-tensorproduct of one-dimensional quadrature rules or product rule. Wendt et al. [92] proposed a projection approach to compute the probability of the output constraint satisfaction in the space of the uncertain inputs, the full grid method was ued for numerical mulivariate intergration. After that, this method was studied by many researchers for instance by Arellano-Garcia and Wozny [96] for monotonicity analysis and Flemming et al. [96] for optimization of closed-loop systems under uncertainty.

In opposition, sparse-grid approaches, based on fully-symmetric integration formulas, need very few integration nodes [97]. They are found to provide an efficient evaluation of high dimensional integrations by reducing computation time significantly. Sparse-grids were first proposed by Smolyak in 1963 [98] and have been recently applied to many fields of numerical computation such as stochastic partial differential equations [99], isogeometric analysis [100], aerodynamics [101], multiscale simulation of polymeric fluids [102], and compositional flow simulations [103].

Quasi-Monte-Carlo (QMC) techniques have been used widely for the caculation of high dimensional integrals [104], [105]. Since the integration rules can be constructed for a large class of underlying weight functions, and the good convergence in the presence of discontinuities, they have been applied widely in many fields [106]–[108].

Finally, in order to evaluate the integrals it is necessary to solve the model equations using for one of the methods for instant Newton (also Newton-Raphson), approximate methods such as artificial neural networks, Fourier series [109], [95].

With respect to large-scale limit and shakedown analysis , in this thesis, stochastic shakedown problems are converted into equivalent deterministic problems using the *direct computation method*. After converting probabilitic constraints into equivalent deterministic ones, the solution of equivalent deterministic shakedown problems can be obtained by nonlinear programming methods with the same numerical efforts as an deterministic problem.

1.3 The achievements of the thesis

The following points may be considered as the new contributions of this thesis:

1. With this approach it is possible for the first time to make safety decisions with direct plasticity methods on a statistical basis corresponding to large-scale problems
2. For the first time, the stochastic shakedown problem with lognormal distribution of strength has been solved sucessfully by a CCOPT approach.
3. Three new efficient algorithms have been developed successfully to solve problems of shakedown analysis of structures under stochastic conditions:
   - Kinematic algorithm (A1) can be used to treat shakedown problems in following situations:
     - Both strength and load are deterministic.
     - The load acting on structure is deterministic, only the strength is distributed normally or lognormally.
     - The strength is deterministic, the load is normally distributed.
     - The strength is random with normal distribution and the load is random with normal distribution.
     - The strength is random with lognormal distribution and the load is random with normal distribution.
   - Dual algorithm (A2) is employed to compute simultaneously upper bound and lower bound shakedown loads in the situations:
     - Both strength and load are deterministic.
     - The load acting on structure is deterministic, the strength of the structure is distributed normally or lognormally.
   - Dual algorithm (A3) is employed to compute simultaneously upper bound and lower bound shakedown loads of Kirchhoff-Love plates subjected to bending in the situations:
     - Both strength and load are deterministic.

- The load acting on plate is deterministic, the strength of the plate is distributed normally or lognormally.

1.4 Organization of the thesis (outline)

The remainder of this thesis is organized as follows. Chapter 2 contains the concept of direct plasticity methods including basic elements of plasticity theory and the theory of limit and shakedown analysis. Chapter 3 is the main contribution of the thesis. This chapter brings an argument for the formulation of stochastic shakedown analysis based on lower bound and upper bound approaches using CCOPT. The smoothed finite element method (ES-FEM) is the numerical method to discretize the problems. For the numerical solution, this chapter presents two algorithms A1 and A2 to solve the stochastic shakedown problems. Numerical applications are investigated in chapter 4 to illustrate the conceptual power and numerical efficiency of the proposed method. In chapter 4, the algorim A3 for shakedown problem of Kirchhoff-Love plates is also presented. Chapter 5 presents briefly the First Order Reliability Method (FORM). The aim of this chapter is to investigate the relation between the CCOPT method and FORM. A conclusion and an outlook to further work can be found in Chapter 6.

Chapter 2

Structural analysis with direct plasticity methods

2.1 Plasticity

This section describes some of the fundamental elements of the theory of plasticity. These elements include yield conditions, plastic flow rule, principle of maximum plastic work and Drucker's stability postulate, power of plastic dissipation. These concepts form the foundations of the theory of plasticity. A good review of these fundamental concepts can be found in [110]–[113].

2.1.1 Inelastic behavior of materials

The modeling of the inelastic behavior of solids is a complex and wide ranging subject. In this section, we present few basic facts about the stress-strain relations for *elastic-perfectly-plastic* and *rigid-perfectly-plastic* materials which are commonly used in mechanical and civil engineering structures. For the purposes of structural analysis, one idealizes the stress-strain diagrams of the materials as shown in Figure 2.1a, which presents the *elastic-perfectly plastic* behavior. In this model, the material behaves elastically below the yield stress and will begin to yield if the stress intensity reaches the yield stress. Stresses are not allowed to become higher than this limit. Moreover, the plastic deformations are many time lager than the elastic limit strains, and in that case the material behavior may be further simplified as *rigid-perfectly plastic* (Figure 2.1b). It can be proved that elastic characteristics do not affect the plastic collapse limit state and thus the application of the elastic perfectly plastic material model becomes same to that of the rigid perfectly plastic model for limit analysis.

In the context of the small-strain theory the total strain tensor $\boldsymbol{\varepsilon}$ is assumed to be decomposed additively into an elastic or reversible part $\boldsymbol{\varepsilon}_e$ and an irreversible part $\boldsymbol{\varepsilon}_p$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p \tag{2.1}$$

The elastic part of the strain is defined by Hooke's law

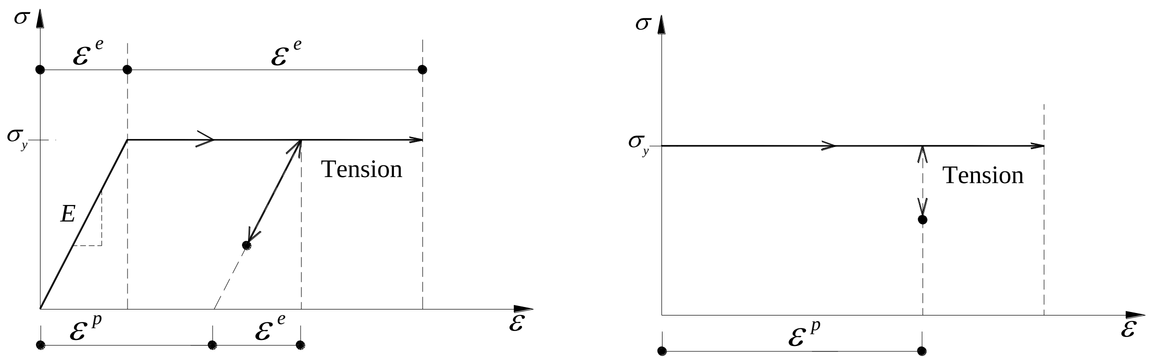
$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}_e \tag{2.2}$$

where $\mathbf{E}$ is the elastic stiffness tensor (which is a fourth-order tensor) with components E_{ijkl}.

The plastic strain rate obeys an associated flow law

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda} \frac{\partial f(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \tag{2.3}$$

where $\dot{\lambda}$ is a non-negative plastic multiplier and $f(\boldsymbol{\sigma}) = 0$ represents a time-independent yield surface.



a) Elastic perfectly plastic (b) Rigid-perfectly plastic

Figure 2. 1 Material models

2.1.2. Yield criteria for isotropic materials

The *yield criterion*, in general written as $f(\boldsymbol{\sigma}) = 0$, defines the stress state for which the material exhibits plastic flow. The set of stress states that satisfy the yield condition forms the so-called yield surface in the stress face. The sign of the expression defining the yield function can always be selected such that $f(\boldsymbol{\sigma}) < 0$ corresponds to *elastic* stress states, $f(\boldsymbol{\sigma}) = 0$ corresponds to *plastic* stress states, and stress states for which $f(\boldsymbol{\sigma}) > 0$ cannot be supported by perfectly plastic material. All the stress states for which $f(\boldsymbol{\sigma}) \leq 0$ are called *plastically admissible*. A yield condition of isotropic materials depends only on the invariants of the stress tensor. In terms of the invariants, the yield condition can be written as

$$f(I_1, J_2, J_3) = 0 \tag{2.4}$$

here $I_1 = \mathrm{tr}\,\boldsymbol{\sigma}$ is the linear invariant of the stress tensor, $J_2 = \frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'$ and $J_3 = \det \boldsymbol{\sigma}'$ are the quadratic and cubic deviatoric stress invariants, respectively. We can list here many yield criteria for isotropic materials such as: von Mises criterion, Tresca criterion, Gurson criterion, Drucker-Prager criterion, Mohr-Coulomb criterion, Rankine criterion. In this thesis, only the von Mises yield condition will be concerned, other criteria are discussed in detail in [111].

von Mises criterion

The von Mises yield function is defined by

$$f(J_2) = \sigma_{eq} - \sigma_y = \left(\frac{3}{2} \boldsymbol{\sigma}' : \boldsymbol{\sigma}' \right)^{1/2} - \sigma_y = \sqrt{3J_2} - \sigma_y \tag{2.5}$$

Where σ_{eq} is the effective stress, sometime call equivalent stress, σ_y is the yield stress of material. In terms of principal stresses, the effective stress is defined as

$$\sigma_{eq} = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]^{1/2} \tag{2.6}$$

In terms of direct and shear stresses:

$$\sigma_{eq} = \left[\frac{3}{2}\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{31}^2\right)\right]^{1/2} \tag{2.7}$$

the effective stress and plastic strain rate can be written as

$$\sigma_{eq} = \left(\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'\right)^{1/2} \tag{2.8}$$

$\boldsymbol{\sigma}'$ is the deviatoric stress given by

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} \tag{2.9}$$

The von Mises function can be written in terms of the second stress invariant, J_2

$$f = \sqrt{3J_2} - \sigma_y \tag{2.10}$$

The second stress invariant is defined as $J_2 = \frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'$ and it is for this reason that plastic flow base on the von Mises yield condition is often referred to as J_2 plasticity. Fig. 2.2 shows the von Mises yield condition for plane stress problem.

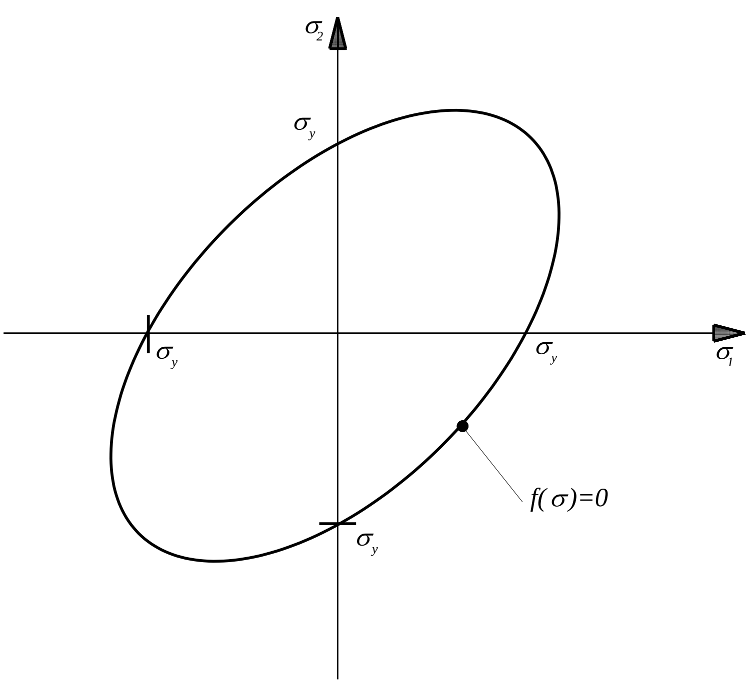


Figure 2. 2 von Mises yield criteria in plane stress

Flow theory of plasticity

Flow rule

The flow rule presents a mathematical description of the evolution of the plastic strain in the course of the load history of the body. The total strain tensor is decomposed into a sum of two symmetric tensors, the elastic (reversible) strain $\boldsymbol{\varepsilon}_e$ and the plastic (irreversible) strain $\boldsymbol{\varepsilon}_p$ as seen in (2.1) . The material is assumed to exhibit plastic flow, the elastic stiffness of the material remains unchanged. This mean the relationship between the elastic strain and the stress is the same as in linear elasticity , see equation (2.2). As long as the stress remains inside the elastic domain, the deformation process is purely elastic and the plastic strain does not change. When the stress state reaches the yield surface, plastic flow is initiated. During plastic flow, the stress state remains on the yield surface, and the yield condition $f(\boldsymbol{\sigma}) = 0$ must be satisfied. The yield condition only provides one additional equation for the evaluation of the plastic strain, which has six independent components. Therefore, an additional rule governing the evolution of plastic flow must be postulated. The flow rule should be deduced from experimental observations. From the theoretical point of view, it is convenient to work with a rule that preserves the validity of the postulate of maximum plastic dissipation (von Mises, 1928,[114]) which plays a key role in the proofs of the fundamental theorems of limit and shakedown analyses.

Postulate of maximum plastic dissipation

Let $\dot{\boldsymbol{\varepsilon}}_p$ be a given plastic strain rate. Among all plastically admissible stress states $\boldsymbol{\sigma}^*$, the power $\boldsymbol{\sigma}^* : \dot{\boldsymbol{\varepsilon}}_p$ is maximized by the actual stress, $\boldsymbol{\sigma}$. This statement can be mathematically written as

$$D(\dot{\boldsymbol{\varepsilon}}_p) \equiv \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p = \max_{f(\boldsymbol{\sigma}^*)\leq 0} (\boldsymbol{\sigma}^* : \dot{\boldsymbol{\varepsilon}}_p) \tag{2.11}$$

where $D(\dot{\boldsymbol{\varepsilon}}_p)$ is the dissipation power per unit volume.

We may interpret equation (2.11) as the condition that the projection of the admissible stress states onto the direction of plastic strain rate be maximized by the actual stress state. This condition holds if

1. the elastic domain is *convex*
2. the direction of the plastic strain rate is *normal* to the yield surface.

These two requirements will be referred to as *convexity* and *normality*. They characterize the so-called *standard materials.* This expression was first used by Radenkovic [115].

Normality rule

In plasticity, the normality rule states that the plastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}_p$ must be normal to the yield surface at smooth points (Figure 2.3c) or must lie between adjacent normals at a corner (non-smooth point) as in figure 2.3d. Figure 2.3a illustrates the violation of the postulate of maximum plastic dissipation due to the lack of normality. Figure 2.3b illustrates the implications of non-convexity of the elastic domain. Since the yield surface is a graphical representation of $f(\boldsymbol{\sigma})$, the direction normal to the yield surface (at a point where the surface is smooth) is determined by the gradient of f, the normality rule may be represented as

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}. \tag{2.12}$$

The *normality rule* is also called the *associated flow rule*. The symbol stand for a scalar multiplier (called the *plastic multiplier*) that controls the magnitude of the plastic strain. Figure 2.3a illustrates the violation of the postulate of maximum plastic dissipation because it is lack of normality. Similarly, figure 2.3b describes the non-convexity of the elastic domain. Convexity and normality are satisfied in figure 2.3c. Figure 2.3d shows that the direction of plastic flow is not uniquely defined by the normality condition at a vertex of the yield surface. In this case, the postulate of maximum plastic dissipation is satisfied for any flow direction lying in the fan between the normal to the 'tangent from the left' and 'tangent from the right' to the yield surface. Such a fan is called the normal cone[111]

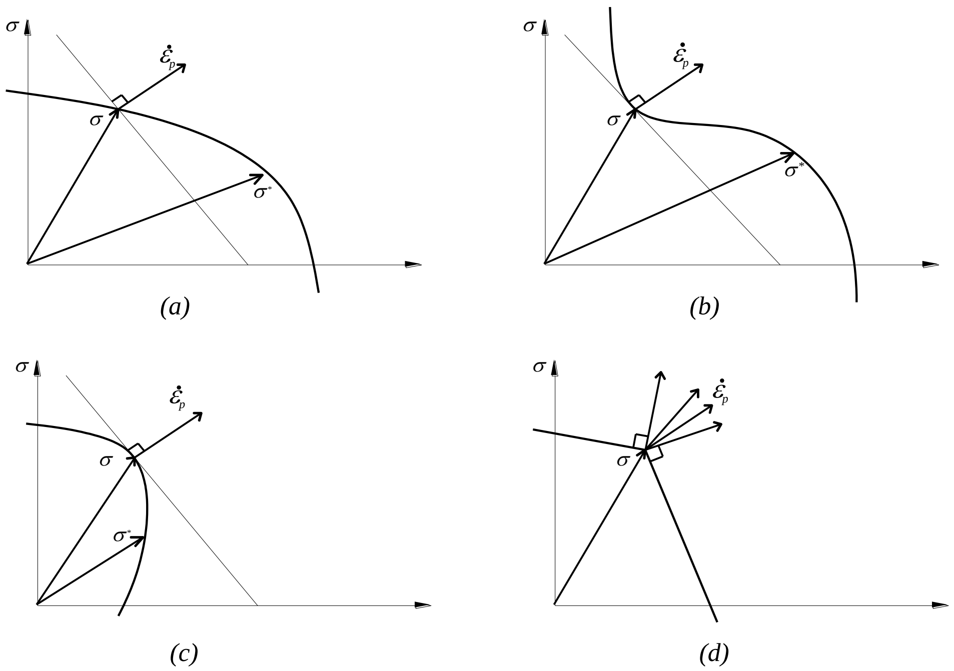


Figure 2. 3 Illustration of convexity and normality.

Convexity of the yield surface

From the postulate of maximum plastic dissipation, the second requirement is the convexity of the yield surface. This has a very important role in plasticity. It permits the use of convex programming tools in limit and shakedown analysis.

Plastic dissipation function

The plastic dissipation function is defined by (2.11)

$$D^p = \max(\boldsymbol{\sigma}^* : \dot{\boldsymbol{\varepsilon}}_p) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p$$

The plastic dissipation for the von Mises criterion and associated flow rule is given by Lubliner [113]

$$D^p(\dot{\boldsymbol{\varepsilon}}_p) = \sqrt{\frac{2}{3}}\sigma_y \sqrt{\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\varepsilon}}_p}\ . \tag{2.13}$$

2.2 Limit analysis of structures

2.2.1 Introduction

Consider a structure made of elastic-perfectly plastic or rigid-perfectly plastic material and there is a set of forces $\mathbf{F}$ acting on it. A common assumption is that all the components of the set of forces change proportionally to a certain load parameter α. This case is referred to as proportional loading. In matrix notation we can write:

$$\mathbf{F} = \alpha \mathbf{F}_0 \tag{2.14}$$

where $\mathbf{F}_0$ is some fixed reference load vector.

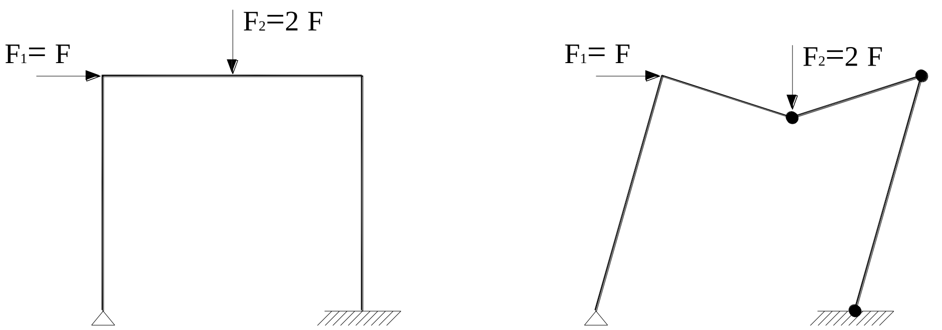


Figure 2. 4 Portal frame with failure mechanism

If the value of α remains sufficiently low, response of the structure is elastic. As α increases and reaches a special value, the first point in the body reaches the plastic state. This state of stress is called elastic limit. Further increase of α will lead to the expansion of plastic region in the structure. The structure gradually forms a collapse mechanism. At limit state, the structure is collapsed by applied forces. The value $\alpha = \alpha_{\lim}$ corresponding to the plastic collapse state is called the safety factor of the structure or the limit load factor.

Limit state of structure can be determined by 'step-by-step' methods. In 'step-by-step' methods the loading path is divided into sufficiently small loading increments with a full analysis of the evolution of stresses and strains. Besides the question of running time, the disadvantage of these methods is that the exact knowledge of the loading history is necessary, which is not realistic in many technical situations. The theory of limit analysis offers a method to solve directly the problem of evaluating the ultimate load-carrying capacity of the structure in which one is primarily interested in the final stage of the plastic response-plastic collapse. It is turns out that the plastic limit load can be assessed without analysing the entire history of the response. This is subject of limit analysis.

2.2.2 Statically and kinematically admissible states

Let us specify the reference loading $\mathbf{F}_0 = (\bar{\mathbf{b}}, \bar{\mathbf{t}})$ by given body forces $\bar{\mathbf{b}}$ and given surface tractions $\bar{\mathbf{t}}$. The principle of virtual work states the static equilibrium of a stress field $\boldsymbol{\sigma}$ which is in equilibrium with $\mathbf{F}_0 = (\bar{\mathbf{b}}, \bar{\mathbf{t}})$

$$\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}V = \int_V \alpha \bar{\mathbf{b}} \cdot \mathbf{u} \, \mathrm{dV} + \int_{S_t} \alpha \bar{\mathbf{t}} \cdot \mathbf{u} \, \mathrm{dS} \, .$$

It is important to note that the kinematic quantities $\mathbf{u}, \boldsymbol{\varepsilon}$ are kinematically compatible by $\boldsymbol{\varepsilon} = \frac{1}{2}\left[(\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^T\right]$ but otherwise arbitrary and have no causal relation with the static quantity $\boldsymbol{\sigma}$. The principle holds also for the time derivatives (virtual power):

$$\int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p \, \mathrm{d}V = \int_V \alpha \bar{\mathbf{b}} \cdot \dot{\mathbf{u}} \, \mathrm{dV} + \int_{S_t} \alpha \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \, \mathrm{dS} \, .$$

For continuous fields and mild conditions the principle is equivalent with the local form of the equlibirium conditions:

$$\begin{aligned} \nabla \boldsymbol{\sigma}(\mathbf{x}, t) &= -\mathbf{b} \quad && \text{in} \, V \\ \mathbf{n} \boldsymbol{\sigma}(\mathbf{x}, t) &= \mathbf{t} \quad && \text{on } S_t \end{aligned}$$

A statically admissible state is described by a stress field $\boldsymbol{\sigma}$ and a load multiplier α^- such that (s.t.)

$$\begin{cases} -\boldsymbol{\sigma} \cdot \nabla = \alpha^- \bar{\mathbf{b}} & \text{in } V \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \alpha^- \bar{\mathbf{t}} & \text{on } S_t \\ f(\boldsymbol{\sigma}) \leq 0 & \text{in } V \end{cases} \tag{2.15}$$

A *kinematically admissible state* is described by a displacement rate field $\dot{\mathbf{u}}$ and a plastic strain rate field $\dot{\boldsymbol{\varepsilon}}$ such that

$$\begin{cases} \dot{\boldsymbol{\varepsilon}} = (\nabla \dot{\mathbf{u}})_{sym} & \text{in } V \\ \dot{\mathbf{u}} = \mathbf{0} & \text{on } S_u \\ \int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}} \, \mathrm{dV} + \int_V \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \, \mathrm{dS} > 0 \end{cases} \tag{2.16}$$

2.2.3 Power equality

During the collapse the external load $\alpha \mathbf{F}_0$ is constant, it does some work on the increasing displacement $\dot{\mathbf{u}}$. The external $\dot{W}_{ext}$ power is the product of the force $\alpha \mathbf{F}_0 = \alpha(\bar{\mathbf{b}}, \bar{\mathbf{t}})$ and the displacement rate in the sense of $\mathbf{.F}_0$., $\dot{W}_{ext} = \int_V \alpha \bar{\mathbf{b}} \cdot \dot{\mathbf{u}} \, \mathrm{d}V + \int_S \alpha \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \, \mathrm{d}S$. This power is supplied to the structure and assuming a steady-state collapse with no inertial effects, it must be dissipated by plastic processes during yielding. This fact can be mathematically written as the *power equality*

$$D_{\mathrm{int}} = \dot{W}_{ext} \tag{2.17}$$

or

$$\int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p \, \mathrm{d}V = \int_V \alpha \bar{\mathbf{b}} \cdot \dot{\mathbf{u}} \, \mathrm{d}V + \int_S \alpha \bar{\mathbf{t}} \cdot \dot{\mathbf{u}} \, \mathrm{d}S \tag{2.18}$$

2.2.4 Theorems of Limit Analysis

- Lower bound theorem states as follows:

If a stress field $\boldsymbol{\sigma}$ can be found which satisfies the statically admissible state (2.18) then the corresponding multiplier α^- cannot exceed the limit multiplier $\alpha_{\lim}$.

- Upper bound theorem states:

Any multiplier α^+ corresponding to a kinematically admissible state (2.19) is not less than the limit multiplier $\alpha_{\lim}$.

Limit analysis is not restricted to proportional loading $\alpha \mathbf{F}_0 = \alpha(\bar{\mathbf{b}}, \bar{\mathbf{t}})$. Any monotonic load path gives the same limit load.

2.2.5 Methods of Limit Analysis

There are two basic approaches to limit analysis corresponding to the two above theorems. *The static approach* is based on the lower bound theorem, according to which the safety factor can be obtained by looking for the maximum statically admissible load multiplier. This task lead to solving a maximum nonlinear optimization problem

$$\begin{aligned} &\alpha_{\lim} = \max \alpha^- \\ &\text{s.t.:} \begin{cases} -\boldsymbol{\sigma} \cdot \nabla = \alpha^- \bar{\mathbf{b}} & \text{in } V \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \alpha^- \bar{\mathbf{t}} & \text{on } S_t \\ f(\boldsymbol{\sigma}) \le 0 & \text{in } V \end{cases} \end{aligned} \tag{2.19}$$

Constraints in (2.22) are the Cauchy equations of equilibrium, static boundary conditions and conditions of plastic admissibility, respectively.

The second one is the *kinematic approach* which is based on upper bound theorem, according to which the safety factor can be obtained by searching for the minimum kinematically admissible load multiplier. This can be achieved by solving the minimum optimization

$$\alpha_{\lim} = \min \alpha^{+}$$
$$\alpha^{+} = \frac{\int_V D(\dot{\boldsymbol{\varepsilon}})\, dV}{\int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\, \mathrm{dV} + \int_V \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\, \mathrm{dS}} \tag{2.20}$$
$$\text{s.t.:} \begin{cases} \dot{\boldsymbol{\varepsilon}} = \left(\nabla \dot{\mathbf{u}}\right)_{sym} & \text{in}\, V \\ \dot{\mathbf{u}} = \mathbf{0} & \text{on } S_u \\ \int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\, \mathrm{dV} + \int_V \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\, \mathrm{dS} > 0 \end{cases}$$

where $\int_V D(\dot{\boldsymbol{\varepsilon}})\, dV$ is the *plastic dissipation power*, the constraints (2.20) are the strain-displacement relations, kinematic boundary conditions, and the condition of positive external power.

We can restrict our attention to solutions with normalized external power by the condition

$$\int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\, \mathrm{dV} + \int_V \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\, \mathrm{dS} = 1 \tag{2.21}$$

Then problem (2.20) can be rewritten as

$$\alpha_{\lim} = \min \int_V D(\dot{\boldsymbol{\varepsilon}})\, dV$$
$$\text{s.t.:} \begin{cases} \dot{\boldsymbol{\varepsilon}} = \left(\nabla \dot{\mathbf{u}}\right)_{sym} & \text{in}\, V \\ \dot{\mathbf{u}} = \mathbf{0} & \text{on } S_u \\ \int_V \bar{\mathbf{b}} \cdot \dot{\mathbf{u}}\, \mathrm{dV} + \int_V \bar{\mathbf{t}} \cdot \dot{\mathbf{u}}\, \mathrm{dS} = 1 \end{cases} \tag{2.22}$$

2.3 Shakedown analysis

2.3.1 Introduction

In section 2.2, we discussed the problem of limit analysis at which all the components of the set of forces acting upon the structures change monotonically. In practice, however, certain types of loads on structures are far from monotonic. For instance, the structure in figure 2.4 may first be subjected to a vertical dead load F_2 and subsequently to alternating lateral wind forces F_1 that vary without any relation to the vertical load. Moreover, the load acting on the structures may be repeated (cyclic) many times or varying arbitrarily in a certain load domain. Types of these loads produce a certain kind of plastic deformation in the structures, which may lead to collapse failure even if the loads remain inside the elastoplastic domain, as we now explain.

Figure 2.5 show the famous so-called Bree diagram[116], which can be used to describe the response of a thin pipe subjected to varying mechanical loading by internal pressure and thermal loading.

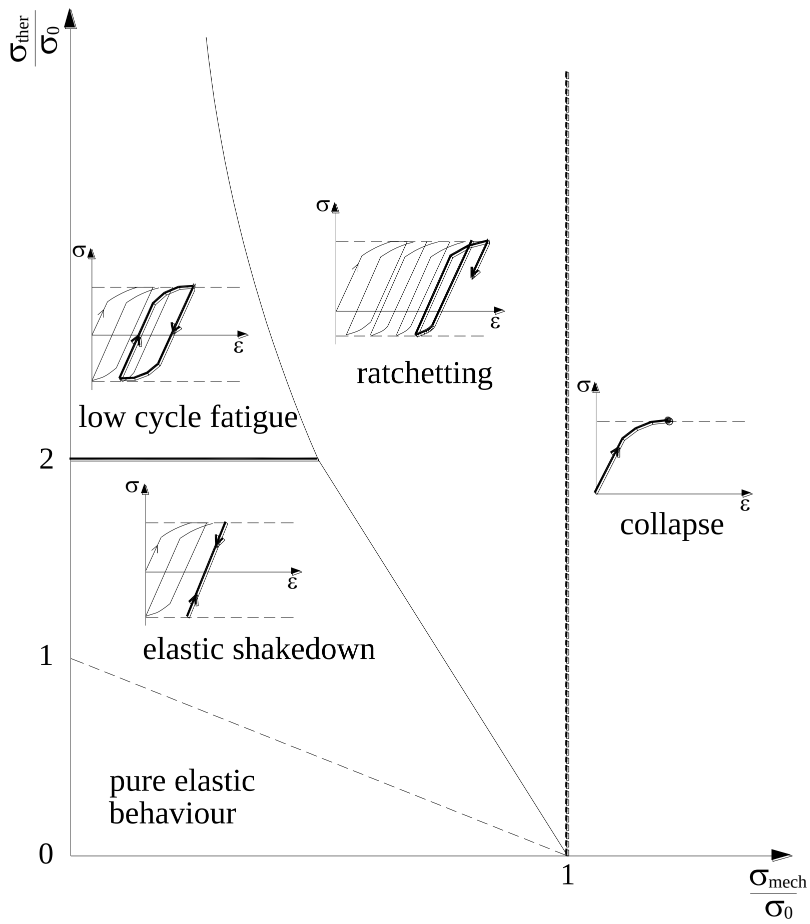


Figure 2. 5Bree Diagram [22], [61], [116]

This diagram divides safe regimes from failure regimes:

1. If the load intensities remain sufficiently low, the structural response is perfectly elastic.
2. As the load intensities become higher than the elastic limit, plastic deformation occurs. It may happen that, after some plastic deformation in the initial load cycles, the structural behaviour becomes eventually elastic. Such stabilization of plastic deformations is called (*elastic*) *shakedown* or *adaptation*.
3. If the stabilization of plastic deformation is not established, the plastic strain increments in each load cycle and after a sufficient number of cycles (and therefore displacements) become so large that the structure becomes unserviceable. This phenomenon is called *incremental collapse* or *ratchetting*.
4. It may happen that the strain increments change sign in every cycle, they tend to cancel each other out and the total deformation remains small (this is so-called *alternating plasticity*). After a sufficient number of cycles, the structure fails. This kind of collapse, due to *alternating plasticity*, is called the *low-cycle fatigue*. Experiments show that alternating

plastic deformations always lead to failure of the material after a relatively small number of cycles, say 100 cycles.

5. If the load intensities become higher than the instantaneous load-carrying capacity of the structure, unconstrained flow mechanism develops. This results in a collapse of the structure.

It is worth noting that the phenomena of incremental collapse and alternating plasticity (low-cycle fatigue) may appear simultaneously.

The main problem of shakedown theory is to investigate whether or not a given structure will shake down under given loads

2.3.2 Description of loading domain

We assume that there are n external loads $F_k(t)$, varying with time t, acting on a structure. Each of the loads can vary independently in the range described by the parameters $\mu_k^{\min}$ and $\mu_k^{\max}$.

$$\mu_k^{\min} F_0 \le F_k(t) \le \mu_k^{\max} F_0 \tag{2.23}$$

These load span a convex polyhedral load domain Ω with $m = 2^n$ corners in the n-dimensional loading space. The load domain Ω can be described as follows.

$$\Omega = \left\{ F(\mathbf{x},t) \middle| F(\mathbf{x},t) = \sum_{k=1}^{n} \mu_k(t) F_0(\mathbf{x}),\ \forall \mu_k(t) \in \left[\mu_k^{\min}; \mu_k^{\max} \right] \right\} \tag{2.24}$$

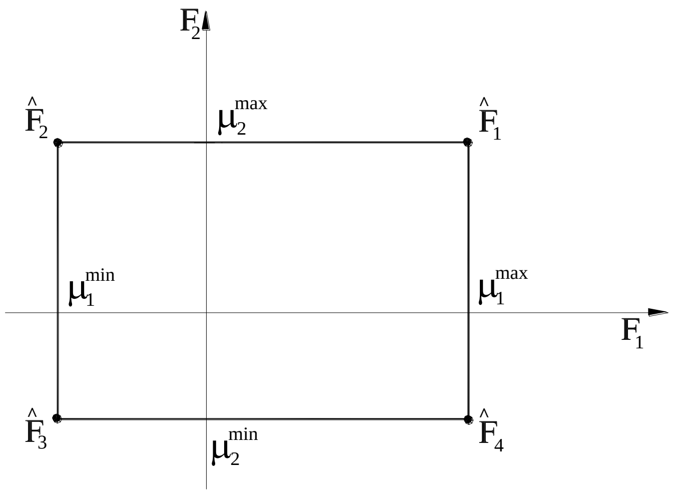


Figure 2. 6 Two dimensional load domain Ω

In many cases it is useful to describe this load domain in the generalized stress space. By denoting the elastic solution $\boldsymbol{\sigma}_k^e$ due to the single load $F_0(\mathbf{x})$ we can define the domain of the elastic stresses associated with in the form

$$\mathbf{S}_e = \left\{ \boldsymbol{\sigma}_e(\mathbf{x},t) = \sum_{k=1}^{n} \mu_k(t) \boldsymbol{\sigma}_{ek}(\mathbf{x}),\ \forall \mu_k(t) \in \left[\mu_k^{\min}; \mu_k^{\max} \right] \right\} \tag{2.25}$$

The closed set $\mathbf{S}_e$ is a convex polytope presenting the envelope of the elastic stresses. It collects the elastic stresses produced by load paths $F(\mathbf{x},t) \in \Omega$. Note that the time is assumed here as an evolution variable, because we always consider the dynamic inertia effects due to the external loads as negligible.

Limit analysis is included as the case of only one load corner, $n = 1$.

2.4.3 Theorem of Shakedown Analysis

Melan's static shakedown theorem (Theorem T1)

Melan's static theorem provides a lower bound to the shakedown load factor[41], [117]. In this work, we restrict ourselves to elastic-perfectly plastic, time-independent material behavior without consideration of hardening. The total stress $\boldsymbol{\sigma}(\mathbf{x},t)$ at a point $x \in V$ of the structure with volume V at time t can be defined as a combination of elastic stress $\boldsymbol{\sigma}_e(\mathbf{x},t)$ and a residual stress $\boldsymbol{\rho}(\mathbf{x},t)$.

$$\boldsymbol{\sigma}(\mathbf{x},t) = \boldsymbol{\sigma}_e(\mathbf{x},t) + \boldsymbol{\rho}(\mathbf{x},t) \tag{2.26}$$

$\boldsymbol{\sigma}_e(\mathbf{x},t) = \mathbf{D}_e\boldsymbol{\varepsilon}_e$ is the stress field which would appear in a fictitious infinitely elastic reference structure under the same load conditions as the original one:

$$\begin{aligned} \nabla \boldsymbol{\sigma}_e(\mathbf{x},t) &= -\mathbf{b} \quad && \text{in } V \\ \mathbf{n}\boldsymbol{\sigma}_e(\mathbf{x},t) &= \mathbf{t} \quad && \text{on } S_t \end{aligned}$$

The residual stress field $\boldsymbol{\rho}(\mathbf{x},t)$ is caused by the evolution of plastic strains. Since $\boldsymbol{\sigma}(\mathbf{x},t)$ and $\boldsymbol{\sigma}_e(\mathbf{x},t)$ are in equilibrium with the same loading , the residual stresses $\boldsymbol{\rho}(\mathbf{x},t)$ satisfy the homogeneous static equilibrium and boundary conditions

$$\begin{aligned} \nabla \boldsymbol{\rho}(\mathbf{x},t) &= \mathbf{0} \quad && \text{in } V \\ \mathbf{n}\boldsymbol{\rho}(\mathbf{x},t) &= \mathbf{0} \quad && \text{on } S_t \end{aligned} \tag{2.27}$$

The total strains

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p = \mathbf{E}^{-1} : \boldsymbol{\sigma} + \boldsymbol{\varepsilon}_p = \mathbf{E}^{-1} : \boldsymbol{\sigma}_e + \mathbf{E}^{-1} : \boldsymbol{\rho} + \boldsymbol{\varepsilon}_p$$

are not compatible, i.e. they cannot be integrated to a displacement field. However, the total strain can be additively decomposed into two compatible fields (in the presense of thermal strains $\boldsymbol{\varepsilon}_{th}$ the elastic strains are not compatible but $\mathbf{E}^{-1} : \boldsymbol{\sigma}_e + \boldsymbol{\varepsilon}_{th}$ would be compatible):

$$\boldsymbol{\varepsilon}_e = \mathbf{E}^{-1} : \boldsymbol{\sigma}_e \text{ and } \boldsymbol{\varepsilon}^* = \mathbf{E}^{-1} : \boldsymbol{\rho} + \boldsymbol{\varepsilon}_p .$$

One criterion for an elastic, perfectly plastic material to shake down elastically is that the plastic strain and therefore the residual stresses become stationary for given loads in a load domain Ω :

$$\lim_{t\to\infty} \dot{\boldsymbol{\varepsilon}}_p(\mathbf{x},t) = 0,$$
$$\lim_{t\to\infty} \dot{\boldsymbol{\rho}}(\mathbf{x}) = 0, \quad \forall\ \mathbf{x} \in V$$

(2.28)

Melan's shakedown theorem can be stated as follows:

If there exists a factor α *and a time-independent self-equilibrated residual stress field* $\bar{\boldsymbol{\rho}}(\mathbf{x})$ *with* $\int_V \bar{\boldsymbol{\rho}} : \mathbf{E} : \bar{\boldsymbol{\rho}}\, \mathrm{d}V < \infty$*, such that for any loading path in load domain* Ω *at any time* t *and in any point* $\mathbf{x}$ *of the body, the yield condition is satisfied*

$$f\left[\alpha\,\boldsymbol{\sigma}_e(\mathbf{x},t) + \bar{\boldsymbol{\rho}}(\mathbf{x})\right] \le 0, \quad \forall x \in V,\ \forall t \tag{2.29}$$

then the structure will shake down elastically under the given load domain.

Proof: We show the proof for perfectly plastic materials and consider the time variant functional

$$J = \frac{1}{2}\int_\Omega [\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}] : \mathbf{E} : [\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}]\, d\Omega\ .$$

Without loss of generality it can be assumed that $\boldsymbol{\rho}(\mathbf{x}, t=0) = 0$. $J > 0$ since $\mathbf{E}$ is positive definite. $\mathbf{E}$ is also symmetric so that the time derivative (rate) is

$$\dot{J} = \int_\Omega [\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}] : \mathbf{E} : \dot{\boldsymbol{\rho}}\, d\Omega\ .$$

A selfequilibrating stress such as $\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}$ does no virtual work with the strain $\boldsymbol{\varepsilon}^* = \mathbf{E}^{-1} : \boldsymbol{\rho} + \boldsymbol{\varepsilon}_p$,

$$\int_V [\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}] : \left(\mathbf{E}^{-1} : \boldsymbol{\rho} + \boldsymbol{\varepsilon}_p\right) \mathrm{d}V = 0 \text{ and } \int_V [\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}] : \left(\mathbf{E}^{-1} : \dot{\boldsymbol{\rho}} + \dot{\boldsymbol{\varepsilon}}_p\right) \mathrm{d}V = 0\ .$$

Combiningt the two last equations

$$\dot{J} = -\int_\Omega [\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}] : \dot{\boldsymbol{\varepsilon}}_p\, d\Omega = -\int_\Omega \left[(\boldsymbol{\sigma}_e + \boldsymbol{\rho}) - (\boldsymbol{\sigma}_e + \bar{\boldsymbol{\rho}})\right] : \dot{\boldsymbol{\varepsilon}}_p\, d\Omega = -\int_\Omega \left[\boldsymbol{\sigma} - (\boldsymbol{\sigma}_e + \bar{\boldsymbol{\rho}})\right] : \dot{\boldsymbol{\varepsilon}}_p\, d\Omega \le 0$$

With Drucker's postulate rate $\dot{J}$ is not positive because the stress $\boldsymbol{\sigma}_e + \bar{\boldsymbol{\rho}}$ is admissible. Therrfotr the functional is bounded from below and monotonously decreasing, so that it converges towards a constant. Therefore also the residual stress converges towards a time invariant residual stress field $\bar{\boldsymbol{\rho}}(\mathbf{x})$

$$\lim_{t\to\infty} \boldsymbol{\rho}(\mathbf{x},t) = \bar{\boldsymbol{\rho}}(\mathbf{x}) \text{ and thus } \lim_{t\to\infty} \dot{\boldsymbol{\varepsilon}}_p(\mathbf{x},t) = \mathbf{0}\ .$$

This follows from the necessary shakedown conditions.

To finish the proof we need to show that the plastic dissipation is finite. From the necessary condition $f\left[\alpha\,\boldsymbol{\sigma}_e(\mathbf{x},t) + \bar{\boldsymbol{\rho}}(\mathbf{x})\right] \le 0$ and Drucker's postulate it follows (König [121]):

$$\left[\boldsymbol{\sigma} - \alpha\left(\boldsymbol{\sigma}_e + \bar{\boldsymbol{\rho}}\right)\right] : \dot{\boldsymbol{\varepsilon}}_p \ge 0$$

or

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p \le \frac{\alpha}{\alpha - 1}(\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}) : \dot{\boldsymbol{\varepsilon}}_p \ge 0 .$$

From this follows for the plastic dissipation

$$\begin{aligned} D_p &= \int_0^\infty \int_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p dV dt \le \frac{\alpha}{\alpha - 1} \int_0^\infty \int_V (\boldsymbol{\rho} - \bar{\boldsymbol{\rho}}) : \dot{\boldsymbol{\varepsilon}}_p dV dt \\ &= -\frac{\alpha}{\alpha - 1} \int_0^\infty \dot{J}(t) dt \le \frac{\alpha}{\alpha - 1} J(0) = \frac{\alpha}{2\alpha - 2} \int_V \bar{\boldsymbol{\rho}} : \mathbf{E}^{-1} : \bar{\boldsymbol{\rho}} dV < \infty \end{aligned}$$

Therefore the dissipation is bounded and the structure shakesdown under the the given load domain. Q.E.D.

The proof shows that a structure cannot shakedown if it contains a singular fictitious elastic stress $\boldsymbol{\sigma}_e(\mathbf{x}, t)$ because then the time independent residual stress field $\bar{\boldsymbol{\rho}}(\mathbf{x})$ must also be singular which contradicts the condition $\int_V \bar{\boldsymbol{\rho}} : \mathbf{E}^{-1} : \bar{\boldsymbol{\rho}} dV < \infty$. This is different in the limit load theorem which only requires that a residual stress field $\boldsymbol{\rho}$ exsist which superimposed on the fictitious elastic stress $\boldsymbol{\sigma}_e$ makes the stress field $\boldsymbol{\sigma}_e + \boldsymbol{\rho}$ adimissible. The condition $\int_V \bar{\boldsymbol{\rho}} : \mathbf{E}^{-1} : \bar{\boldsymbol{\rho}} dV < \infty$ is not required so that limit analysis can be successfully applied on crack containing structures [ư]

The greatest value α^- which satisfies the theorem is call shakedown load factor. The static shakedown theorem is formulated in terms of stresses and gives a lower bound to α^-. This leads to the mathematical optimization problem

$$\begin{aligned} &\alpha^- = \max \alpha \\ &\text{s.t.:} \begin{cases} f\left[\alpha \boldsymbol{\sigma}_e(\mathbf{x}, t) + \bar{\boldsymbol{\rho}}(\mathbf{x})\right] \le 0 & \forall \mathbf{x} \in V \\ \nabla \bar{\boldsymbol{\rho}}(\mathbf{x}) = \mathbf{0} & \forall \mathbf{x} \in V \\ \bar{\boldsymbol{\rho}}(\mathbf{x})\mathbf{n} = \mathbf{0} & \forall \mathbf{x} \in S_t \end{cases} \end{aligned} \tag{2.30}$$

Kinematic shakedown theorem by Koiter (Theorem T2)

Using a plastic strain field to formulate a shakedown criterion, the kinematic shakedown theorem is the counterpart of the static one. The theorem was given by Koiter [20] and some of its applications in analysis of incremental collapse were derived by Gokhfeld [118], Sawczuk ([119], [120]). Same as proposed by Koiter for plastic strain field, we introduce here an admissible cycle of plastic generalized strain field $\Delta \boldsymbol{\varepsilon}_p$. The plastic generalized strain rate $\dot{\hat{\boldsymbol{\varepsilon}}}^p$ may not necessarily be compatible at each instant during the period T but the plastic generalized strain accumulation over the period:

$$\Delta \boldsymbol{\varepsilon}_p = \int_0^T \dot{\boldsymbol{\varepsilon}}_p dt \tag{2.31}$$

is required to be kinematically compatible such that

$$\begin{aligned} \Delta \boldsymbol{\varepsilon}_p &= \nabla \left[(\Delta \mathbf{u}) \right]_{sym} \\ \Delta \mathbf{u} &= 0 \quad \text{on } S_u \end{aligned} \tag{2.32}$$

and

$$\int_V \int_0^T \boldsymbol{\sigma}_e : \dot{\boldsymbol{\varepsilon}}_p dt dV > 0 \tag{2.33}$$

where $\mathbf{u}(\mathbf{x})$ denotes the displacement vector in the structure.

Kinematic theorem (Theorem T2):

1. *Shakedown may happen if the following inequality is satisfied*

$$\int_0^T \int_V \boldsymbol{\sigma}_e(\mathbf{x},t) : \dot{\boldsymbol{\varepsilon}}_p dV dt \le \int_0^T \int_V D^p(\dot{\boldsymbol{\varepsilon}}_p) dV dt \tag{2.34}$$

2. *Shakedown cannot happen when the following inequality holds*

$$\int_0^T \int_V \boldsymbol{\sigma}_e(\mathbf{x},t) : \dot{\boldsymbol{\varepsilon}}_p dV dt > \int_0^T \int_V D^p(\dot{\boldsymbol{\varepsilon}}_p) dV dt \tag{2.35}$$

Based on the kinematic theorem, an upper bound of the shakedown limit load multiplier α^+ can be computed. The shakedown problem can be seen as a mathematical minimization problem in nonlinear programming

$$\begin{aligned} &\alpha^+ = \min \frac{\int_0^T \int_V D^p(\dot{\boldsymbol{\varepsilon}}_p) dV dt}{\int_0^T \int_V \boldsymbol{\sigma}_e(\mathbf{x},t) : \dot{\boldsymbol{\varepsilon}}_p dV dt} \\ &\text{s.t.:} \begin{cases} \Delta \boldsymbol{\varepsilon}_p = \int_0^T \dot{\boldsymbol{\varepsilon}}_p dt & \\ \Delta \boldsymbol{\varepsilon}_p = \nabla \left[(\Delta \mathbf{u}) \right]_{sym} & \text{in } V \\ \Delta \mathbf{u} = 0 & \text{on } S_u \end{cases} \end{aligned} \tag{2.36}$$

In order to calculate the shakedown limit load multiplier, the two following methods can be applied: separated and unified methods. While the former analyses the two different failure modes (incremental plasticity (ratchetting) and alternating plasticity) separately, the latter analyses them simultaneously. Both methods deserve special attention due to their role in structural computation.

2.4.4 Separated shakedown limit

As was mentioned above, incremental collapse and alternating plasticity may occur simultaneously. These in-adaptation modes can be defined precisely in the following way (König 1987,[121])

1. A perfect incremental collapse process (over a certain time interval $(0,T)$ *) is a process of plastic deformation* $\bar{\boldsymbol{\varepsilon}}(\mathbf{x},t)$ *in which a kinematically admissible plastic generalized strain increment* $\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x},t) = \bar{\boldsymbol{\varepsilon}}(\mathbf{x},T) - \bar{\boldsymbol{\varepsilon}}(\mathbf{x},0)$ *is attained in a proportional and monotonic way, namely*

$$\begin{aligned}
&\Delta\bar{\boldsymbol{\varepsilon}} = \nabla\left[\dot{\mathbf{u}}(\mathbf{x})\right]_{sym} \\
&\dot{\mathbf{u}} = 0 \quad \text{on } S_u \\
&\dot{\bar{\boldsymbol{\varepsilon}}} = \dot{\Lambda}(\mathbf{x},t)\cdot\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x}) \\
&\dot{\Lambda}(\mathbf{x},t) \geq 0 \\
&\Lambda(\mathbf{x},0) = 0 \\
&\Lambda(\mathbf{x},T) = 1
\end{aligned} \tag{2.37}$$

2. An alternating plasticity process is any process of plastic deformation $\boldsymbol{\varepsilon}(\mathbf{x},t)$ *within a certain time interval* $(0,T)$ *such that the total increment of the plastic generalized strain* $\Delta\boldsymbol{\varepsilon}(\mathbf{x})$ *over this period is zero,*

$$\Delta\boldsymbol{\varepsilon}(\mathbf{x}) = \int_0^T \dot{\boldsymbol{\varepsilon}}(\mathbf{x},t)\,dt = 0 \tag{2.38}$$

The criteria of safety with respect to alternating plasticity or incremental collapse may be obtained by substituting the plastic strain history (2.37) or (2.38) into the shakedown condition (2.34).

From the definitions (2.37) and (2.38), it is easy to see that any plastic generalized strain history $\boldsymbol{\varepsilon}_p(\mathbf{x},t)$, which leads to a kinematically admissible plastic generalized strain increment within a periodic interval $(0,T)$, can be decomposed into two components:perfectly incremental collapse and alternating plasticity process, see König [121].

Incremental collapse criterion

If the safety condition against any form of perfectly incremental collapse is sought, the plastic strain field is assumed by (2.37). On substituting (2.37) into (2.34), one obtains

$$W_{ex} = \int_0^T\int_V \boldsymbol{\sigma}_e(\mathbf{x},t)\dot{\Lambda}(\mathbf{x},t)\cdot\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})\,dV\,dt \leq W_{in} = \int_0^T\int_V D^p(\dot{\Lambda}\Delta\bar{\boldsymbol{\varepsilon}})\,dV\,dt \tag{2.39}$$

On account of the properties of the dissipation function and the plastic strain history (2.37) we can write

$$W_{in} = \int_0^T \int_V D^p(\dot{\Lambda}\Delta\bar{\boldsymbol{\varepsilon}})\,dV\,dt = \int_0^T \dot{\Lambda} \int_V D^p(\Delta\bar{\boldsymbol{\varepsilon}})\,dV\,dt = \int_V D^p(\Delta\bar{\boldsymbol{\varepsilon}})\,dV \tag{2.40}$$

From (2.39), the smallest upper bound of the incremental collapse limit could be attained when the external work W_{ex} assumes its maximum and the internal dissipation W_{in} takes its minimum. To this end, the function $\dot{\Lambda}(\mathbf{x},t)$ is selected in such a way that $\dot{\Lambda}(\mathbf{x},t) \neq 0$ only when the product $\boldsymbol{\sigma}_e(\mathbf{x},t)\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ takes its maximu $\alpha\int_V \bar{\boldsymbol{\sigma}}_e(\mathbf{x})\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})\,dV \leq \int_V D^p(\Delta\bar{\boldsymbol{\varepsilon}})\,dV$ m possible value for a given load domain Ω. In this case, the external power W_{ex} can be written as

$$W_{ex} = \int_0^T \int_V \boldsymbol{\sigma}_e(\mathbf{x},t)\dot{\Lambda}(\mathbf{x},t)\cdot\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})\,dV\,dt = \int_V \bar{\boldsymbol{\sigma}}_e(\mathbf{x})\cdot\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})\,dV \tag{2.41}$$

where

$$\bar{\boldsymbol{\sigma}}_e(\mathbf{x})\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x}) = \max\left(\boldsymbol{\sigma}_e(\mathbf{x},t)\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})\right) \tag{2.42}$$

$$\mathbf{S}_e = \left\{\boldsymbol{\sigma}_e(\mathbf{x},t) = \sum_{k=1}^{n} \mu_k(t)\boldsymbol{\sigma}_{ek}(\mathbf{x}),\ \forall\mu_k(t)\in\left[\mu_k^{\min};\mu_k^{\max}\right]\right\} \tag{2.43}$$

Therefore the safety condition with respect to any form of perfectly incremental collapse is as follows

$$\alpha\int_V \bar{\boldsymbol{\sigma}}_e(\mathbf{x})\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})\,dV \leq \int_V D^p(\Delta\bar{\boldsymbol{\varepsilon}})\,dV \tag{2.44}$$

for any kinematically admissible $\Delta\bar{\boldsymbol{\varepsilon}}$.

If the load domain is prescribed by (2.25) the safety condition (2.44) becomes

$$\alpha\int_V \sum_{k=1}^{n} \bar{\mu}_k J_k(\mathbf{x})\,dV \leq \int_V D^p(\Delta\bar{\boldsymbol{\varepsilon}})\,dV \tag{2.45}$$

Here

$$J_k(\mathbf{x}) = \boldsymbol{\sigma}_{ek}(\mathbf{x})\Delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$$

$$\bar{\mu}_k = \begin{cases}\mu_k^{\max} & \text{if} \quad J_k(\mathbf{x}) \geq 0\\ \mu_k^{\min} & \text{if} \quad J_k(\mathbf{x}) < 0\end{cases} \tag{2.46}$$

From (2.45) the shakedown load multiplier α^+ with respect to incremental collapse can be formulated as a non-linear programming problem:

$$\alpha^+ = \min\left(\frac{W_{in}}{W_{ex}}\right) \tag{2.47}$$

or in normalized form

$$\begin{aligned} &\alpha^{+} = \min W_{in} \\ &\text{s.t.:} \quad W_{ex} = 1 \end{aligned} . \tag{2.48}$$

Alternating plasticity criterion

As it was mentioned previously, if we consider the safety condition against alternating plasticity then the plastic strain field must be satisfied (2.38). The shakedown condition (2.34) in this case has the form

$$\alpha \int_0^T \int_V \boldsymbol{\sigma}_e(\mathbf{x},t)\,\dot{\boldsymbol{\varepsilon}}_p(\mathbf{x},t)\,dV\,dt \le \int_0^T \int_V D^p(\dot{\boldsymbol{\varepsilon}}_p)\,dV\,dt \tag{2.49}$$

With

$$\alpha > 1, \quad \int_0^T \hat{\dot{\boldsymbol{\varepsilon}}}(\mathbf{x},t)\,dt = 0 \quad \text{for every } \mathbf{x} \in \mathrm{V} . \tag{2.50}$$

Starting from the kinematic theorem and the last constraint in (2.50), the optimization problem leading to the most stringent limit condition can be established at each point $\mathbf{x}$ separately. Let us assume that the safety factor α corresponding to a point $\mathbf{x}$ is $\alpha(\mathbf{x})$. The global safety factor against the alternating plasticity is $\min_V \alpha(\mathbf{x})$. It is easy to see that the safety factor $\alpha(\mathbf{x})$ can be determined by solving the following constrained optimization problem [121]

$$\begin{aligned} &\frac{1}{\alpha(\mathbf{x})} = \max \int_0^T \boldsymbol{\sigma}(\mathbf{x},t)\,\dot{\boldsymbol{\varepsilon}}_p(\mathbf{x},t)\,dt \\ &\text{s.t.:} \quad \begin{cases} \int_0^T D^p(\dot{\boldsymbol{\varepsilon}}_p)\,dt = 1 \\ \int_0^T \dot{\boldsymbol{\varepsilon}}_p(\mathbf{x},t)\,dt = 0 \end{cases} \end{aligned} \tag{2.51}$$

By solving this problem, the static shakedown condition against any form of alternating plasticity can be obtained.

A given structure is safe against alternating plasticity if there exists a time-independent generalized stress field $\boldsymbol{\rho}$ *which, if superimposed on the envelope of elastic generalized stresses, does not violate the yield condition*

$$f\left(\boldsymbol{\sigma}_e(\mathbf{x},t) + \boldsymbol{\rho}\right) \le 0 . \tag{2.52}$$

It should be noted that the stress field $\boldsymbol{\rho}$ in (2.52) is an arbitrary time-independent generalized stress field and not necessarily self-equilibrated as that in Melan's theorem. If we define a general stress response

$$\boldsymbol{\sigma}^*(\mathbf{x}) = \sum_{k=1}^{n} (\bar{\mu}_k + \mu_k)\boldsymbol{\sigma}_{ek}(\mathbf{x}) \tag{2.53}$$

where $\boldsymbol{\sigma}_{ek}(\mathbf{x})$ is the elastic generalized stress field in the reference structure when subjected to the k^{th} load and

$$\bar{\mu}_k = \frac{\mu_k^{\max} + \mu_k^{\min}}{2}, \quad |\mu_k| = \frac{\mu_k^{\max} - \mu_k^{\min}}{2} \tag{2.54}$$

From (2.51), the plastic shakedown load factor (lower bound) may be computed:

$$\alpha = \min_{\mathbf{x}} \frac{1}{F\left(\boldsymbol{\sigma}^*(\mathbf{x}) + \boldsymbol{\rho}(\mathbf{x})\right)} \tag{2.55}$$

where

$$f = F - 1. \tag{2.56}$$

The sign of μ_k must be chosen so that the value of the function F^* is maximum. By considering the alternating characteristic of the stress corresponding to an alternating strain rate, the optimal time-independent generalized stress field $\boldsymbol{\rho}$ can be defined by

$$\boldsymbol{\rho} = -\sum_{k=1}^{n} \bar{\mu}_k \boldsymbol{\sigma}_{ek}. \tag{2.57}$$

Then the plastic shakedown limit load multiplier can be finally represented as

$$\alpha = \min_{\mathbf{x}} \frac{1}{F\left(\sum_{k=1}^{n} \mu_k \boldsymbol{\sigma}_{ek}(\mathbf{x})\right)} \tag{2.58}$$

or in the following equivalent form:

$$\alpha = \min_{\mathbf{x}} \frac{1}{F\left[\frac{1}{2}\sum_{k=1}^{n} \pm\left(\mu_k^{\max} - \mu_k^{\min}\right)\boldsymbol{\sigma}_{ek}(\mathbf{x})\right]} \tag{2.59}$$

where all the combinations of the signs $\pm$ must be accounted for.

2.4.5 Unified shakedown limit

In practical computations, in most cases it is impossible to apply Melan's and Koiter's theorems to find directly the shakedown limit defined by the minimum of incremental plasticity limit and alternating plasticity limit. The difficulty here is the presence of the time-dependent generalized stress field $\boldsymbol{\sigma}_e(\mathbf{x},t)$ in (2.32). These obstacles can be overcome with the help of the following two convex-cycle theorems, introduced by König and Kleiber [122].

Theorem T3:

"Shakedown will happen over a given load domain Ω if and only if it happens over the convex

envelope of Ω *".*

Theorem T4:

"Shakedown will happen over any load path within a given load domain Ω *if it happens over a cyclic load path containing all vertices of* Ω *".*

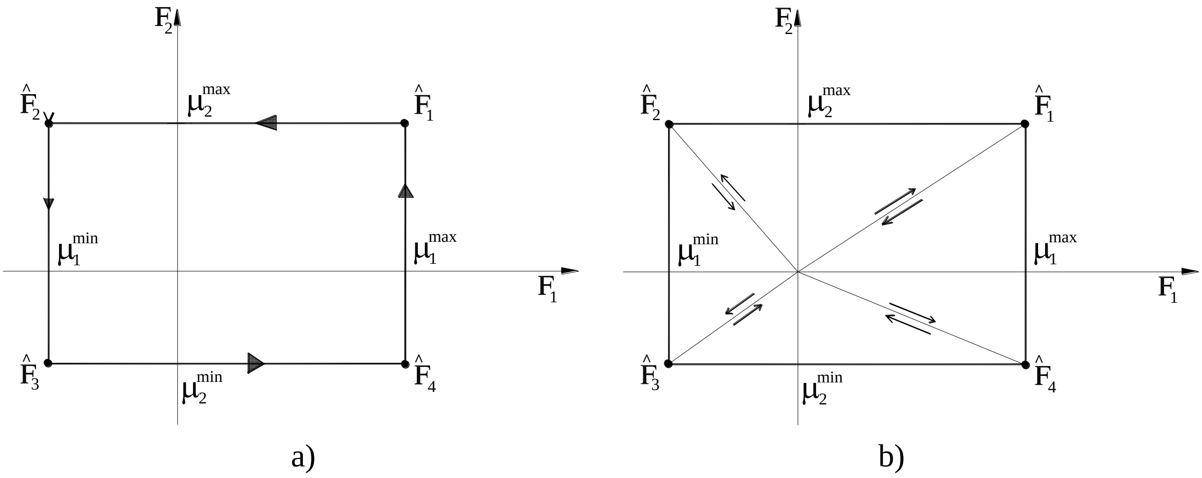


Figure 2. 7 Critical cycles of load for shakedown analysis

These theorems, which hold for convex load domains and convex yield surfaces, permit us to consider one cyclic load path instead of all loading history. They allow us to examine only the stress and strain rate fields at every vertex of the given load domain instead of computing an integration over the time cycle. Based on these theorems, König and Kleiber suggested a load scheme as shown in figure 2.7a for two independently varying loads. This scheme was applied in a simple step-by-step shakedown analysis by Borkowski and Kleiber [123]. Another scheme (figure 2.7.b) was adopted later by Morelle [124]. Extensions and implementations of these theorems can also be found in the works of Morelle[6], [125], Nguyen and Morelle [126], Polizzotto [127], Yan [128].

Let us restrict ourselves to the case of a convex polyhedral load domain Ω . The question is how to apply the above theorems to eliminate the time-dependent elastic generalized stress field $\boldsymbol{\sigma}_e(\mathbf{x},t)$ and time integrations in the lower and upper shakedown theorems. In order to do so, let us consider a special load cycle $(0,T)$ passing through all vertices of the load domain Ω such as

$$\mathbf{F}(\mathbf{x},t) = \sum_{k=1}^{m} \delta(t_k)\hat{\mathbf{F}}_k(\mathbf{x}) \tag{2.60}$$

where $m = 2^n$ is the total number of vertices of Ω , n is the total number of varying loads, $\delta(t_k)$ is the Dirac distribution with the property

$$\delta(t_k) = \begin{cases} 1 & \text{if} \quad t = t_k \\ 0 & \text{if} \quad t \neq t_k \end{cases} \tag{2.61}$$

Over this load path, the strain at any instant t is represented by

$$\boldsymbol{\varepsilon}(t) = \sum_{k} \delta(t_k)\dot{\boldsymbol{\varepsilon}}_k \tag{2.62}$$

At each instant (or at each load vertex), the kinematical condition may not be satisfied, however the accumulated generalized strain over a load cycle

$$\Delta\boldsymbol{\varepsilon} = \sum_{k=1}^{m} \dot{\boldsymbol{\varepsilon}}_k \tag{2.63}$$

must be kinematically compatible.

Obviously, the Melan condition required in the whole load domain will be satisfied if and only if it is satisfied at all vertices (or the above special loading cycle) of the domain due to the convexity property of the load domain and the yield function. This remark permits us to replace the time-dependent generalized stress field $\boldsymbol{\sigma}_e(\mathbf{x},t)$ by its values calculated only at load vertices. We have the following static shakedown theorem.

Theorem T5:

The necessary and sufficient condition for shakedown to occur is that there exists a permanent residual generalized stress field $\overline{\boldsymbol{\rho}}(\mathbf{x})$, statically admissible, such that

$$f\left(\boldsymbol{\sigma}_e(\mathbf{x},\hat{F}_k) + \overline{\boldsymbol{\rho}}(\mathbf{x})\right) \le 0 \qquad \forall k = \overline{1,m}\ . \tag{2.64}$$

The application of loading cycle (2.60) also leads to the elimination of time integration in the kinematic shakedown condition as stated in the theorem hereafter.

Theorem T6:

The necessary and sufficient condition for shakedown to occur is that there exists a plastic accumulation mechanism $\dot{\hat{\boldsymbol{\varepsilon}}}_k$ such that

$$\begin{cases} \int_0^T dt \int_V \boldsymbol{\sigma}_e(\mathbf{x},\hat{P}_k) : \dot{\boldsymbol{\varepsilon}}_p dV \le \int_0^T dt \int_V D^p(\dot{\boldsymbol{\varepsilon}}_p) dV \\ \Delta\boldsymbol{\varepsilon} = \sum_{k=1}^{m} \dot{\boldsymbol{\varepsilon}}_k \\ \Delta\boldsymbol{\varepsilon}_p = \nabla\left[(\Delta\mathbf{u})\right]_{sym} \end{cases} \tag{2.65}$$

From (2.30) and (2.36) the bounds of the shakedown limit load multiplier corresponding to the static theorem and the kinematic theorem now can be reformulated in simpler forms

1. *The lower bound*

$$\alpha^{-} = \max \alpha$$
$$\text{s.t.:}\begin{cases} f\left[\alpha\boldsymbol{\sigma}_e(\mathbf{x},\hat{F}_k) + \overline{\boldsymbol{\rho}}(\mathbf{x})\right] \le 0 & \forall \mathbf{x} \in V \\ \nabla\overline{\boldsymbol{\rho}}(\mathbf{x}) = \mathbf{0} & \forall \mathbf{x} \in V \\ \overline{\boldsymbol{\rho}}(\mathbf{x})\mathbf{n} = \mathbf{0} & \forall \mathbf{x} \in S_t \end{cases} \tag{2.66}$$

2. *The upper bound (in normalized form)*

$$\alpha^{+} = \min \int_0^T \int_V D^p(\dot{\boldsymbol{\varepsilon}}_p)\,dV\,dt$$
$$\text{s.t.:}\begin{cases} \Delta\boldsymbol{\varepsilon}_p = \int_0^T \dot{\boldsymbol{\varepsilon}}_p\,dt & \\ \Delta\boldsymbol{\varepsilon}_p = \nabla\left[(\Delta\mathbf{u})\right]_{sym} & \text{in } V \\ \Delta\mathbf{u} = 0 & \text{on } S_u \\ \int_0^T \int_V \boldsymbol{\sigma}_e(\mathbf{x},\hat{P}_k) : \dot{\boldsymbol{\varepsilon}}_p\,dV\,dt = 1 & \end{cases} \tag{2.67}$$

Let us note that if there is only one load and this load increases monotonically, then the according load domain Ω reduces to one point:

$$\mu_1^{\min} = \mu_1^{\max} \tag{2.68}$$

In this case it is easy to see that the above upper bound and lower bound reduce to the formulations of the upper and lower bounds of limit load factor. This means that limit analysis can be considered as a special case of shakedown analysis.

2.4 Deterministic Programming

In this section, deterministic programming are recalled to prepare establishment of stochastic shakedown problems in chapter 3. The concept of deterministic programming includes: lower bound and upper bound discretization with FEM, upper bound formulation with ES-FEM, relationship between upper bound and lower bound of shakedown loads. These concepts can be found in [26], [129], [130].

The lower bound theorem states that the limit and shakedown load calculated from a statically admissible stress field is a lower bound on the true collapse load and shakedown load, respectively. A statically admissible stress field satisfies equilibrium, the stress boundary conditions, and the yield condition. These conditions are formulated in the restrictions of a constrained maximum problem. But the standard displacement FEM do not satisfy the the static conditions in the weak sense. A static FEM formulation with elements with linear variation of the stress and statically admissible stress jumps at interelement boundaries has been suggested to achieve statically admissible stress fields in the weak sense [131]. More recently Ho et al.

[132] have proposed equilibrium cell-based smooth finite element to achieve statically adimissble stress fields for shakedown analysis.

For a good compromise between simplicity and accuracy, a mixed simplex finite element has been proposed for shakedown analysis, with an interpolation of both stresses and displacements based on the classical Prange-Hellinger-Reissner functional (Garcea et a., 2005) [133].

Finding shape functions for stress which satisfy equilibrium was only possible for simple cases. Therefore, FEM for complex problems is done with shape functions for displacements which is much more simple. These displacement finite elements satisfy the conditions of the upper bound that the velocity field is kinematically admissible. Using them also for the lower bound is standard practice for a numerical approximation. The practice is acceptable and will be used here. Some caution is necessary however because the lower bound is not a strict bound for such finite elements.

2.4.1 Lower bound discretization with FEM

Applying the principle of virtual work to the equilibrium equations of the time independent residual stress field $\overline{\boldsymbol{\rho}}$, we get their corresponding ‘’weak form’’ and which can be discretized by means of the finite element method. The lower bound shakedown load factor can be found by solving the following maximized optimization problem:

$$\alpha^{-} = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{B}^T \overline{\boldsymbol{\rho}} = \mathbf{0} \\ f\left[\alpha \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right] \leq \sigma_y , \quad \forall k = \overline{1, m} , \quad \forall i = \overline{1, NG} \end{cases} \tag{2.69}$$

where:

+ $\mathbf{B} = \left[w_1 \mathbf{B}_1, w_2 \mathbf{B}_2, \ldots, w_i \mathbf{B}_i, \ldots, w_{NG} \mathbf{B}_{NG} \right]$

+ $\overline{\boldsymbol{\rho}}^T = \left[\overline{\boldsymbol{\rho}}_1^T, \overline{\boldsymbol{\rho}}_2^T, \ldots, \overline{\boldsymbol{\rho}}_i^T, \ldots, \overline{\boldsymbol{\rho}}_{NG}^T \right]$

+ $\overline{\boldsymbol{\rho}}$ is the discretized residual stress field with its components computed at Gauss points.

+ $\boldsymbol{\sigma}_{ik}^E$ is the fictitious elastic stress vector at Gauss point i corresponding to the vertex $\hat{P}_k$ of the load domain D.

+ $\mathbf{B}_i$ is the deformation matrix $\mathbf{B}(\mathbf{x})$ at Gauss point $\mathbf{x}_i$

+ w_i is the weight factor of the Gauss point i

+ NG denotes the total number of Gauss points of the discretized structure

+ σ_y is the yield stress of the material

2.4.2 Upper bound discretization with FEM

If the von Mises yield condition is used $W_{in} = \sum_{k=1}^{m}\sum_{i=1}^{NG}\sqrt{\frac{2}{3}}\sigma_y\sqrt{\dot{\boldsymbol{\varepsilon}}_{ik}^T\mathbf{D}\,\dot{\boldsymbol{\varepsilon}}_{ik}+\varepsilon_0^2}$, $W_{ex} = \sum_{k=1}^{m}\sum_{i=1}^{NG} w_i\dot{\boldsymbol{\varepsilon}}_{ik}^T\boldsymbol{\sigma}_{ik}^e$ and the discretized formulation of the upper bound problem is:

$$\alpha^+ = \min\sum_{k=1}^{m}\sum_{i=1}^{NG}\sqrt{\frac{2}{3}}\sigma_y\sqrt{\dot{\boldsymbol{\varepsilon}}_{ik}^T\mathbf{D}\,\dot{\boldsymbol{\varepsilon}}_{ik}+\varepsilon_0^2}$$
$$\text{s.t.}: \begin{cases} \sum_{k=1}^{m}\dot{\boldsymbol{\varepsilon}}_{ik}-\mathbf{B}_i\dot{\mathbf{u}}=\mathbf{0} & \forall i=\overline{1,NG} \\ \mathbf{D}_v\dot{\boldsymbol{\varepsilon}}_{ik}=\mathbf{0} & \forall i=\overline{1,NG},\quad \forall k=\overline{1,m} \\ \sum_{k=1}^{m}\sum_{i=1}^{NG} w_i\dot{\boldsymbol{\varepsilon}}_{ik}^T\boldsymbol{\sigma}_{ik}^e-1=0 \end{cases} \tag{2.70}$$

where:

+ $\dot{\boldsymbol{\varepsilon}}_{ik}$ is strain rate vector at Gauss point i, corresponding to load vertex $\hat{P}_k$

$$\dot{\boldsymbol{\varepsilon}}_{ik}=\left[\dot{\varepsilon}_{11}^i,\dot{\varepsilon}_{22}^i,\dot{\varepsilon}_{33}^i,\gamma_{12}^i,\gamma_{23}^i,\dot{\varepsilon}_{31}^i\right]_k^T$$

+ $\boldsymbol{\sigma}_{ik}^e$ is the fictitious elastic stress vector at Gauss point i corresponding to the vertex $\hat{F}_k$
+ $\dot{\mathbf{u}}$ is the displacement rate vector
+ $\mathbf{B}_i$ is the deformation matrix $\mathbf{B}(\mathbf{x})$ at Gauss point $\mathbf{x}_i$
+ w_i is the weight factor of the Gauss point i.
+ NG denotes the total number of Gauss points of the discretized structure
+ σ_y is the yield stress of the material
+ ε_0^2 is the small positive number to avoid the singularity of the dissipation function
+ $\mathbf{D}$, $\mathbf{D}_v$ are square matrices, in a 3D model they have the form:

$$\mathbf{D}=\begin{bmatrix}1&1&1&0&0&0\\1&1&1&0&0&0\\1&1&1&0&0&0\\0&0&0&1/2&0&0\\0&0&0&0&1/2&0\\0&0&0&0&0&1/2\end{bmatrix}\qquad \mathbf{D}_v=\begin{bmatrix}1&1&1&0&0&0\\1&1&1&0&0&0\\1&1&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

2.4.3 ES-FEM discretization

2.4.3.1 Brief introduction to ES-FEM

The edge-based Smoothed Finite Element Method (ES-FEM) is a member of the family of the so-called Smoothed Finite Element Method (SFEM). The general theory of the SFEM can be found in the text book of Liu and Nguyen Thoi [134].There is a number of publications covering different topics concerning S-FEM, like [135]–[141] and many more.

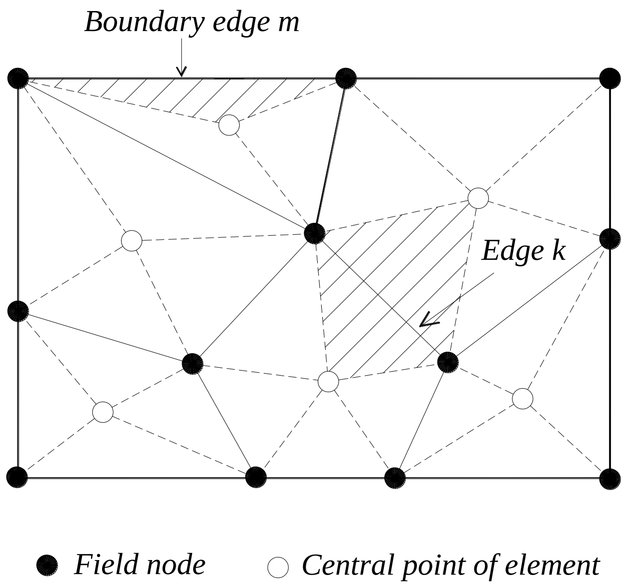


Figure 2. 8 2D smoothing domain creation

In general, one can apply the ES-FEM to a 2D mesh with polygonal elements and T3 elements as depicted in Fig. 2.8. However, the basic of the ES-FEM using T3 element is presented here.

Basically, the ES-FEM-T3 inherits the fundamental properties of FEM-T3 using triangular elements including the triangular mesh discretization, the linear node shape functions and approximated displacement field on the whole problem domain. However, the difference between the standard FEM-T3 and the ES-FEM-T3 isthat he FEM-T3 computes the element stiffness matrix $\mathbf{K}_e$ on the elements while the ES-FEM-T3 uses the gradient smoothing technique [142] to compute the local stiffness matrix $\bar{\mathbf{K}}^{(i)}$ on the so-called edge-based smoothing domains $\Omega^{(i)}$. Based on the mesh of the standard FEM-T3, a smoothing domain (SD) $\Omega^{(i)}$ in the ES-FEM-T3 is created by connecting two endpoints of the edge to the centroids of the adjacent triangles sharing the edge k. With such way, the structural domain contains Ne the smoothing domain $\Omega^{(i)}$ such that $\Omega = \bigcup_{i=1}^{Ne} \Omega^{(i)}$ and $\Omega^{(m)} \cap \Omega^{(n)} = \phi$ if $m \neq n$. Where Ne is the total number of SD, is also the total number of edges of the finite element mesh.

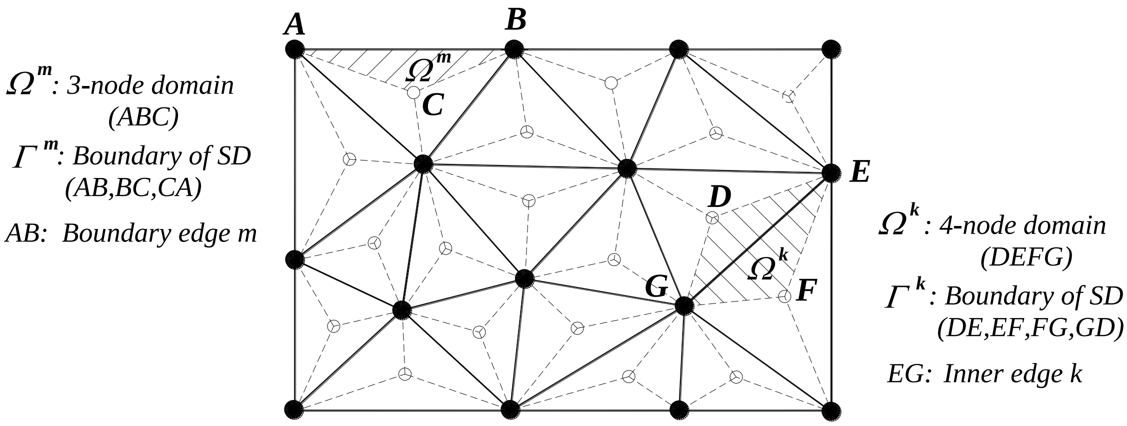


Figure 2. 9 3T element mesh

Using the gradient smoothing technique [134],[142], the compatible strain $\boldsymbol{\varepsilon} = (\nabla \mathbf{u})_{sym}$ is used to define the smoothed strain $\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$:

$$\bar{\boldsymbol{\varepsilon}}_k = \bar{\boldsymbol{\varepsilon}}_k(\mathbf{x}) = \int_{\Omega_k^i} \boldsymbol{\varepsilon}(\mathbf{x})\mathbf{W}_k(\mathbf{x})d\Omega \tag{2.71}$$

where $\bar{\boldsymbol{\varepsilon}}_k$ being the smoothed strain in Ω_k^i, $\mathbf{W}_k(\mathbf{x})$ is a given smoothing function and satisfies at least the unity property as

$$\int_{\Omega_k^i} \mathbf{W}_k(\mathbf{x})d\Omega = 1 \tag{2.72}$$

The local smoothing function

$$\mathbf{W}_k(\mathbf{x}) = \begin{cases} \dfrac{1}{A_k^i}, & x \in \Omega_k^i \\ 0, & x \notin \Omega_k^i \end{cases} \tag{2.73}$$

is used where A_k^i is the area of the k^{th} smoothing domain Ω_k^i.

Due to the fact that the same shape functions are used in the ES-FEM as in the FEM, which leads to the nodal force vector in the ES-FEM evaluated in the same way as in the FEM. The smoothing strains can be written as

$$\bar{\boldsymbol{\varepsilon}}_k = \sum_{I \in N_n^k} \bar{\mathbf{B}}_I(\mathbf{x}_k)\mathbf{u}_I \tag{2.74}$$

where N_n^i is the set of all nodes of the elements that share the edge i, for instant $N_n^i \equiv \{D, E, F, G\}$, $\bar{\mathbf{B}}_I(\mathbf{x}_k)$ is the smoothed strain-displacement matrix on the domain Ω^i

$$\bar{\mathbf{B}}_I(\mathbf{x}_k) = \begin{bmatrix} \bar{B}_{I1}(\mathbf{x}_k) & 0 \\ 0 & \bar{B}_{I2}(\mathbf{x}_k) \\ \bar{B}_{I1}(\mathbf{x}_k) & \bar{B}_{I2}(\mathbf{x}_k) \end{bmatrix} \tag{2.75}$$

where

$$\bar{B}_{\mathrm{Im}}(\mathbf{x}_k) = \frac{1}{A^i}\int_{\Gamma^i} \mathbf{N}_I(\mathbf{x})\mathbf{n}_m^{(i)}(\mathbf{x})d\Gamma \qquad (m = 1, 2) \tag{2.76}$$

Here Γ^i is the boundary of the smoothing domain Ω^i with the area A^i, $\mathbf{n}_m^{(i)}(\mathbf{x})$ is the outward normal vector on the boundary Γ^i.

Using the same assembly manner as in the FEM, the global smoothed stiffness matrix

$$\mathbf{K}^{SFEM} = \sum_{i=1}^{Ne} \bar{\mathbf{K}}^{(i)} \tag{2.77}$$

where $\bar{\mathbf{K}}^{(i)}$ is the smoothed stiffness matrix of the smoothing domain Ω^i and its entries are computed as

$$K_{IJ}^{(i)} = \int_{\Omega^i} \bar{\mathbf{B}}_I^T \mathbf{D} \bar{\mathbf{B}}_J d\Omega = A^{(i)} \bar{\mathbf{B}}_I^T \mathbf{D} \bar{\mathbf{B}}_J . \tag{2.78}$$

2.4.3.2 ES-FEM formulation

Shakedown problems can be formulated with help of ES-FEM. In [26], the ES-FEM using three-node linear triangular elements is applied successfully for deterministic problems of limit and shakedown analysis of structures. By discretizing the entire problem domain into smoothing domains, applying the strain smoothing technique described above and using the von Mises yield condition, the upper bound discretized formulation can be written as:

$$\begin{gathered} \alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{Ne} A^{(i)} \sqrt{\frac{2}{3}} \sigma_y \sqrt{\dot{\bar{\boldsymbol{\varepsilon}}}_{ik}^T \mathbf{D} \, \dot{\bar{\boldsymbol{\varepsilon}}}_{ik} + \varepsilon_0^2} \\ \text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\bar{\boldsymbol{\varepsilon}}}_{ik} - \bar{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall i = \overline{1, Ne} \\ \mathbf{D}_v \dot{\bar{\boldsymbol{\varepsilon}}}_{ik} = \mathbf{0} & \forall i = \overline{1, Ne}, \quad \forall k = \overline{1, m} \\ \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\bar{\boldsymbol{\varepsilon}}}_{ik}^T \boldsymbol{\sigma}_{ik}^e = 1 \end{cases} \end{gathered} \tag{2.79}$$

For the sake of simplicity, some new notations are introduced:

- The new strain rate vector $\dot{\mathbf{e}}_{ik}$:

$$\dot{\mathbf{e}}_{ik} = A^{(i)} \mathbf{D}^{1/2} \dot{\bar{\boldsymbol{\varepsilon}}}_{ik}$$

- The new fictitious elastic stress field $\mathbf{t}_{ik}$:

$$\mathbf{t}_{ik} = \mathbf{D}^{-1/2} \boldsymbol{\sigma}_{ik}^e$$

- The new deformation matrix:

$$\hat{\mathbf{B}}_i = A^{(i)}\mathbf{D}^{1/2}\overline{\mathbf{B}}_i$$

We get a simplified formulation for the upper bound shakedown load factor:

$$\alpha^{+} = \min \sum_{k=1}^{m}\sum_{i=1}^{Ne}\sqrt{\frac{2}{3}}\sigma_y\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}$$
$$\text{s.t.:}\begin{cases}\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik}-\hat{\mathbf{B}}_i\dot{\mathbf{u}}=\mathbf{0} & \forall i=\overline{1,Ne}\\ \mathbf{D}_v\dot{\mathbf{e}}_{ik}=\mathbf{0} & \forall i=\overline{1,Ne},\quad \forall k=\overline{1,m}\\ \sum_{k=1}^{m}\sum_{i=1}^{Ne}\dot{\mathbf{e}}_{ik}^T\mathbf{t}_{ik}-1=0\end{cases} \tag{2.80}$$

2.4.4 Dual relationship between upper bound and lower bound

If $m=1$ the formulation (2.80) reduces to that of the limit analysis problem. Andersen *el al*. [47] while considering a problem of minimizing a sum of Euclidean norms, found that in the case of limit analysis there exists a dual form for problem (2.80). The conclusion of Andersen can be extended for shakedown analysis problems in ([48], [129]):

If there exists a finite solution α^+ for the kinematic shakedown load factor (2.80) *with* $\varepsilon_0^2=0$ *then α^+ has the dual form as follows:*

$$\alpha^{-} = \max_{\alpha,\beta_i,\gamma_{ik}} \alpha$$
$$\text{s.t.:}\begin{cases}\sum_{i=1}^{Ne}\hat{\mathbf{B}}_i^T\boldsymbol{\beta}_i=\mathbf{0}\\ \left\|\boldsymbol{\gamma}_{ik}+\boldsymbol{\beta}_i+\alpha\,\mathbf{t}_{ik}\right\|\le\sqrt{\frac{2}{3}}\sigma_y\end{cases} \tag{2.81}$$

in which $\alpha,\boldsymbol{\beta}_i,\boldsymbol{\gamma}_{ik}$ are Lagrange multipliers, further more $\boldsymbol{\beta}_i,\boldsymbol{\gamma}_{ik}$ can be interpreted as residual and hydrostatic stress vectors related to the compatibility and incompressibility conditions. The multiplier α , of course, represents the lower bound shakedown load factor.

It is proved that the form (2.81) is identical with the discretized form of the lower bound shakedown, which is formulated by Melan's static theorem. Therefore, problem (2.81) can be written in another form as

$$\alpha^{-} = \max \alpha$$
$$\text{s.t.:}\begin{cases}\sum_{i=1}^{Ne}\hat{\mathbf{B}}_i^T\overline{\boldsymbol{\rho}}_i=\mathbf{B}^T\overline{\boldsymbol{\rho}}=\mathbf{0}\\ f\left[\alpha\,\boldsymbol{\sigma}_{ik}^E+\overline{\boldsymbol{\rho}}_i\right]\le\sqrt{\frac{2}{3}}\sigma_y\ ,\quad \forall k=\overline{1,m}\ ,\quad \forall i=\overline{1,Ne}\end{cases} \tag{2.82}$$

Chapter 3

Shakedown Analysis of Structures under Stochastic Conditions with Chance Constrained Programming

3.1 Chance constrained Programming

Stochastic programming deals with optimization problems involving uncertaint parameters for which stochastic models are employed. Optimization problems regarding to stochastic models occur in alomost all areas of science, engineering, finance,etc.. Stochastic programming has many aspects for instant worst-case and distributional robustness analysis, robust optimization, recourse programming and chance constrained optimization (CCOPT). Robust optimization and recourse programming are two additional approaches to optimization under uncertainty, which can be applied in settings slightly diff erent from that of CCOPT. The main purpose of this section is to give a short introduction on the CCOPT. The concept presented here is to mention only the most important facts for reference, since full information can be found some literatures, for instant [109], [143].

In general, a stochastic optimization problem with single chance constraints takes the forms

$$\min_{\mathbf{x}} \mathrm{E}\left[f(\mathbf{x},\boldsymbol{\xi})\right] \qquad (3.1)$$

s.t.

either $$\mathrm{Prob}\left[g_i(\mathbf{x},\boldsymbol{\xi}) \le 0\right] \ge \alpha_i,\ i = 1,\dots,m. \qquad (3.2)$$

or $$\mathrm{Prob}\left[g_i(\mathbf{x},\boldsymbol{\xi}) \le 0,\ i = 1,\dots,m\right] \ge \alpha. \qquad (3.3)$$

where:

- $f, g : R^n \times R^p \to R$ are at least diff erentiable w.r.t. $\mathbf{x} \in X \subset R^n$
- $\mathbf{x}$ is a vector of deterministic variables
- $\boldsymbol{\xi} \subset \Omega \subset R^n$ is a vector of random variables with joint probability density function $\phi(\boldsymbol{\xi})$
- $\mathrm{Prob}[\cdot]$, $\mathrm{E}[\cdot]$ are probability and expectation operators;
- $\mathrm{Prob}\left[g(\mathbf{x},\boldsymbol{\xi}) \le 0\right] \ge \alpha$ is chance (probability) constraints.
- $\alpha \in [0.5\ \ 1]$ is reliablity (probability) level

Constraints (3.2) are single chance constraints while constraints (3.3) are joint chance constraints. The random vector $\boldsymbol{\xi}$ can be either a Gaussian or non-Gaussian.

The CCOPT problems can be calssified as in figure 3.. All possible combinations may occur, e.g., dynamic problems with linear process model and time dependent, non-Gaussian distributed uncertainties.

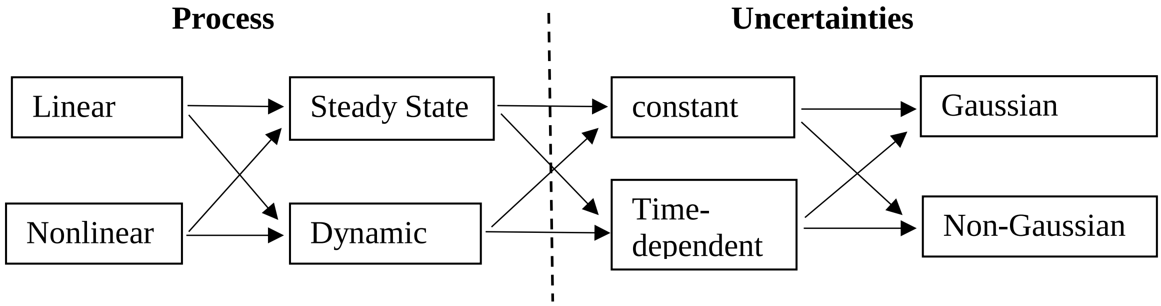


Figure 3. 1 Classification of CCOPT problems

Since all possible combinations of the single components lead to a valid CCOPT problem, 16 different types of optimization problems can be established. Each of the CCOPT problems defined above has unique properties which allow certain techniques to be used in the solution process. In general, the solution process consists of three parts: transforming the probabilistic into deterministic constraints, solution of the model equations, and numerical integration.

There are several approaches to the evaluation of chance constraints, the famous approaches are:

- Direct computation method.
- Linearization methods.
- Projection approaches.
- Analytical Approximation.

They are the most important methods for transforming the probabilistic constraints into deterministic ones. They typically require faster evaluation of multidimensional probability integrals for large-scale problems. This can be achieve through efficient numerical integration techniques full grids, sparse grids, quasi-Monte-Carlo cubature. In order to evaluate the integrals it is necessary to solve the model equations using for one of the methods for instant Newton (also Newton-Raphson), approximate methods such as Artificial Neural Networks, Fourier series. It is clear that the solution of CCOPT problems is computationally very demanding, especially in the presence of a higher number of uncertainties. Investigation of efficient numerical approaches to the solution of such problems is still necessary.

3.2 Probability distributions

There are two the probability distributions used in this thesis. The idea here is to mention only the most important facts for reference, since detailed information can be found in various text books, e.g., in [144].

3.2.1 Gaussian distribution

The normal distribution is one of the most commonly used probability distribution for applications. A normal distribution in a variate x with mean or expectation μ and variance σ^2 is a statistic distribution with the probability density function as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (3.4)$$

The notation $N(\mu, \sigma^2)$ is used for the normal distribution. Random variables X with a normal distribution are said to be a normal random variable and we write $X \sim N(\mu, \sigma^2)$.

Standard normal distribution

The case where $\mu = 0$ and $\sigma = 1$ is called the standard normal distribution. The equation for the standard normal distribution as

$$f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \quad (3.5)$$

The figure 3.1 show the plot of the standard normal probability density function.

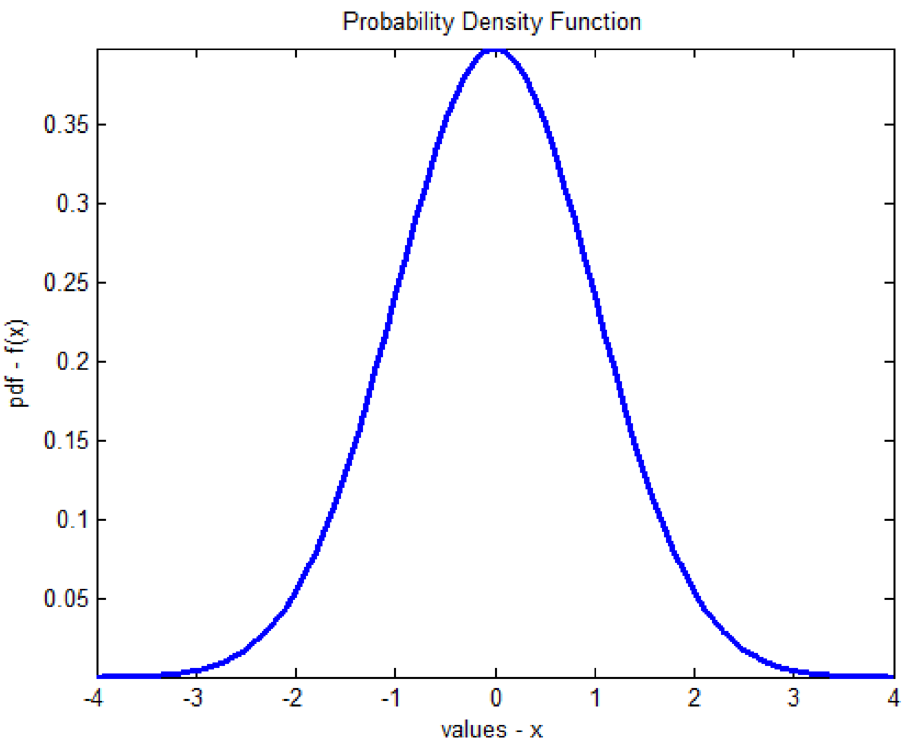


Figure 3. 2 The standard normal probability density function

The shape of a normal distribution looks like a bell, thus it is often called a ''bell curve''. While statisticians and mathematicians uniformly use the term "normal distribution" for this distribution, physicists sometimes call it a Gaussian distribution.

The cumulative distribution function of the standard normal distribution is defined as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \quad (3.6)$$

Some properties of the function $\Phi(x)$:

$$\Phi(-\infty) = 0, \;\; \Phi(+\infty) = 1, \;\; \Phi(x) \text{ is increasing} \tag{3.7}$$

$$\Phi(-x) = 1 - \Phi(x), \;\; \Phi(0) = 0.5 \tag{3.8}$$

The normal distribution is non-zero and symmetric about μ so that μ is also the median. Therefore, the normal distribution may not be a suitable model for variables that can assume only positive values or are otherwise strongly skewed. For such random variables, other distributions, such as the lognormal distribution, may be more suitable.

Yield stress and ultimate stress data is typically assumed to follow a lognormal distribution or a Weibull distribution, [145]. But for simplicity also a normal distribution is used in practice because the normal distribution is an adequate approximation of the lognormal distribution if the standard deviation is sufficiently small.

3.2.2 Lognormal distribution

A lognormal distribution is a probability distribution with a normally distributed logarithm. A random variable is lognormally distributed if its logarithm is normally distributed. The distribution is useful for variables which cannot assume negative values.

The notation $\mathrm{LN}(\mu,\sigma^2)$ is used for the lognormal distribution. Random variables X are said to be lognormally distributed if their logarithm is normally distributed. We write $\ln(X) \sim \mathrm{N}(\mu,\sigma^2)$ or $X \sim \mathrm{LN}(\mu,\sigma^2)$ .The probability density function is defined by the mean μ and standard deviation σ :

$$f(x) = \begin{cases} \dfrac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} & , \quad x > 0 \\ 0 & , \quad x \le 0 \end{cases} \tag{3.9}$$

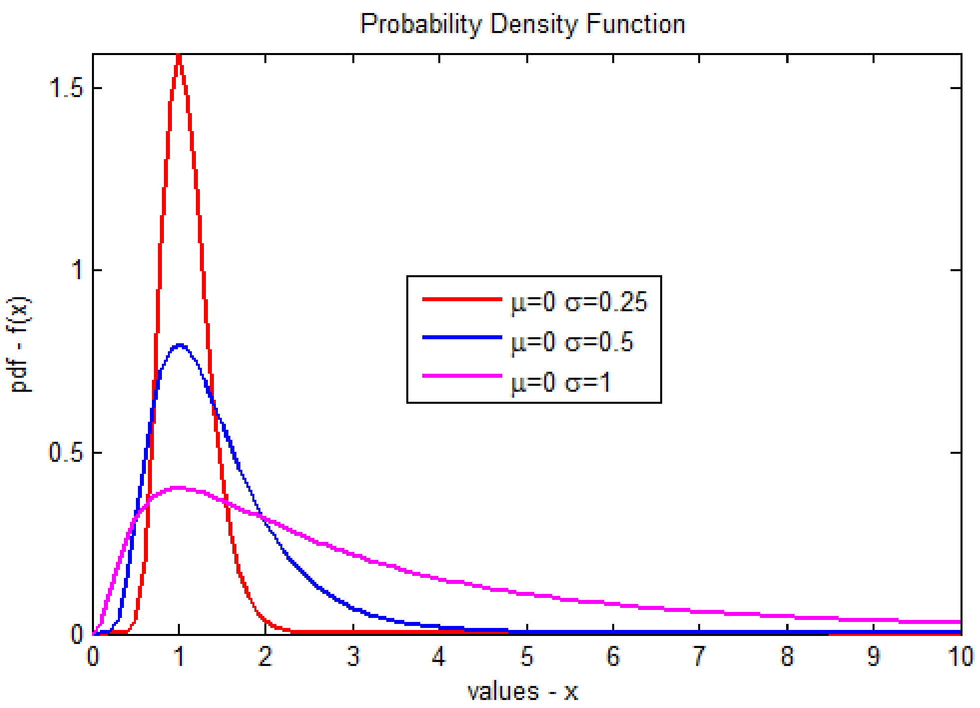


Figure 3. 3 Examples of lognormal probability density function

In (3.9), μ and σ are parameters of the lognormal distribution. The mean m and variance v of a lognormal distribution can be computed as follows

$$\begin{aligned} m &= e^{\mu + \sigma^2/2} \\ v &= e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right) \end{aligned} \tag{3.10}$$

The parameters can be calculated from the mean and the variance.

$$\begin{aligned} \mu &= \ln\left(\frac{m^2}{\sqrt{v + m^2}}\right) \\ \sigma &= \sqrt{\ln\left(\frac{v}{m^2} + 1\right)} \end{aligned} \tag{3.11}$$

The cumulative distribution function of the lognormal distribution is related with the standard normal distribution:

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right), \quad x \geq 0 .$$

3.3 Structures under uncertainty

There are genuine stochastic loadings like wind or the motion of the sea. But even dead loads like weight could be uncertian due to varying dimensions and material composition. Strength is uncertain because of random yield stress and ultimate stress. Typically random load and random strength are stochastically independent because the density of e.g. steel and its yield stress are independent. Strength of rods, beams, plates and shells are random due to random dimensions of cross sections. Therefore, weight and strength could be correlated for structures which are reprenseted by rod, beam, plate or shell finite elements. We will use only continumm elements so that it is justified to assume that loads and strength are stochastically independent.

Wind or the motion of the sea are time variant and best modelled as stochastic process. This is not needed here because shakedown analysis makes the plastic failure problem time invariant.

Material data and geometric variations are best modelled as stochastic fields, [146]. We choose the simpler model of random variables with the same distribution for a structural part made from the nominally same material because the emphasis of the thesis is put on the formulation and solution of lower and upper bounds as chance contrained programs.

3.3.1 Structure under random strength

- **Lower bound approach to chance constrained programming**

Starting from the discretized form (2.82) of the deterministic formulation the lower bound load factor α^- is the maximum of all safe load factors α :

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{0} \\ f\left[\alpha \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right] - r_i \le 0, \quad \forall k = \overline{1,m}, \quad \forall i = \overline{1,Ne} \end{cases} \tag{3.12}$$

Consider the situation that the strength of the material is not given but must be modelled through random variables $r = r(\omega)$ in a certain probability space. Under uncertainty, the inequalities of are not always satisfied, the probability of the i^{th} yield condition being satisfied is required to be greater than some reliability level ψ_i. Problem (3.12) becomes an individually chance constrained programming problem:

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{0} \\ \text{Prob}\left[f\left(\alpha \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right) - r_i(\omega) \le 0 \right] \ge \psi_i \end{cases} \tag{3.13}$$

Let us consider the individual chance constraint:

$$\text{Prob}\left[f\left(\alpha \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right) - r_i(\omega) \le 0 \right] = \text{Prob}\left[f_i - r_i(\omega) \le 0 \right] \ge \psi_i \tag{3.14}$$

The strength distributes normally

We assume that the strength $r_i(\omega)$ of the material follows a Gaussian distribution $N(\mu_i, \sigma_i)$ with mean value μ_i and standard deviation σ_i. Let us transform to a standard normal distribution. The yield condition can be written as $\frac{f_i - \mu_i}{\sigma_i} \leq \frac{r_i(\omega) - \mu_i}{\sigma_i}$ and we have:

$$\mathrm{Prob}\left[f_i \leq r_i(\omega) \right] = \mathrm{Prob}\left[\frac{f_i - \mu_i}{\sigma_i} \leq \frac{r_i(\omega) - \mu_i}{\sigma_i} \right] \tag{3.15}$$

Using the property of the cumulative distribution function (c.d.f.) of the standard normal distribution $\Phi(-x) = 1 - \Phi(x)$, we can write (3.15) as follows:

$$\mathrm{Prob}\left[\frac{f_i - \mu_i}{\sigma_i} \leq \frac{r_i(\omega) - \mu_i}{\sigma_i} \right] = 1 - \Phi\left[\frac{f_i - \mu_i}{\sigma_i} \right] = \Phi\left[\frac{\mu_i - f_i}{\sigma_i} \right]. \tag{3.16}$$

Now the probabilistic condition (3.14) is replaced by

$$\Phi\left[\frac{\mu_i - f_i}{\sigma_i} \right] \geq \psi_i . \tag{3.17}$$

Introducing a new variable $\kappa_i = \Phi^{-1}(\psi_i)$ so that $\psi_i = \Phi(\kappa_i)$, inequality (3.17) becomes:

$$\Phi\left[\frac{\mu_i - f_i}{\sigma_i} \right] \geq \Phi(\kappa_i) . \tag{3.18}$$

Because Φ is monotonic it holds

$$\kappa_i \leq \frac{\mu_i - f_i}{\sigma_i} \qquad \text{or} \qquad f_i \leq \mu_i - \kappa_i \sigma_i . \tag{3.19}$$

Finally we get an equivalent deterministic formulation of the static approach:

$$\begin{aligned} &\alpha^- = \max \alpha \\ &\text{s.t.: } \begin{cases} \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \bar{\boldsymbol{\rho}}_i = \mathbf{0} \\ f\left(\alpha \boldsymbol{\sigma}_{ik}^E + \bar{\boldsymbol{\rho}}_i \right) \leq \mu_i - \kappa_i \sigma_i \qquad , \quad \forall k = \overline{1, m} \ , \quad \forall i = \overline{1, Ne} \end{cases} \end{aligned} \tag{3.20}$$

The strength is distributed lognormally

Consider the i^{th} individual chance constraint:

$$\mathrm{Prob}\left[f\left(\alpha \boldsymbol{\sigma}_{ik}^E + \bar{\boldsymbol{\rho}}_i \right) - r_i(\omega) \leq 0 \right] = \mathrm{Prob}\left[f_i - r_i(\omega) \leq 0 \right] \geq \psi_i \tag{3.21}$$

If the strength of the material $r_i(\omega)$ is distributed lognormally with parameters μ_i and σ_i then $\ln[r_i(\omega)]$ is distributed normally with mean μ_i and standard deviation σ_i, in short $\ln r_i \sim N(\mu_i, \sigma_i^2)$.

The probabilistic constraint (3.21) can be rewritten with the complementary cumulative distribution function:

$$1 - \mathrm{Prob}\left[r_i(\omega) \le f_i\right] \ge \psi_i \tag{3.22}$$

Similar to the case of normally distributed strength, we would like to find an equivalent deterministic of problem (3.13), Let us make some transformations:

$$\begin{aligned}
\mathrm{Prob}\left[r_i(\omega) \le f_i\right] &= \mathrm{Prob}\left[\ln(r_i) \le \ln(f_i)\right] \\
&= \mathrm{Prob}\left[\frac{\ln(r_i) - \mu_i}{\sigma_i} \le \ln\left(\frac{\ln(f_i) - \mu_i}{\sigma_i}\right)\right] \\
&= \Phi\left[\frac{\ln(f_i) - \mu_i}{\sigma_i}\right]
\end{aligned}$$

Using the property of the cumulative distribution function (c.d.f.) of the standard normal distribution $\Phi(-x) = 1 - \Phi(x)$, we can write (3.22) as follows:

$$1 - \Phi\left[\frac{\ln(f_i) - \mu_i}{\sigma_i}\right] = \Phi\left[\frac{\mu_i - \ln(f_i)}{\sigma_i}\right] \ge \psi_i \tag{3.23}$$

Now the probabilistic condition (3.21) is replaced by

$$\Phi\left[\frac{\mu_i - \ln(f_i)}{\sigma_i}\right] \ge \psi_i \tag{3.24}$$

Introducing a new variable $\kappa_i = \Phi^{-1}(\psi_i)$ so that $\psi_i = \Phi(\kappa_i)$, inequality (3.24) becomes:

$$\Phi\left[\frac{\mu_i - \ln(f_i)}{\sigma_i}\right] \ge \Phi(\kappa_i) \tag{3.25}$$

Because Φ is monotonic thus

$$\kappa_i \le \frac{\mu_i - \ln(f_i)}{\sigma_i} \tag{3.26}$$

From (3.26) we have:

$$f_i \le e^{\mu_i - \kappa_i \sigma_i} \tag{3.27}$$

Finally we get an equivalent deterministic formulation of the static approach for lognormally distributed strength:

$$\alpha^- = \max \alpha$$
$$\text{s.t.:} \begin{cases} \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{0} \\ f\left[\alpha \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right] \le e^{\mu_i - \kappa_i \sigma_i}, \quad \forall k = \overline{1,m}, \quad \forall i = \overline{1,Ne} \end{cases} \tag{3.28}$$

- **Upper bound approach to chance constrained programming**

In chapter 2, we presented the deterministic shakedown problem based on Koiter's theorem. The ES-FEM formulation is written in the following normalized form with the von Mises plastic dissipation rate:

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall i = \overline{1,Ne} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} & \forall i = \overline{1,Ne}, \quad \forall k = \overline{1,m} \\ \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 \end{cases} \tag{3.29}$$

If the strength r_i is an uncertain quantity, the objective function of the kinematic problem is a stochastic variable and problem (3.29) becomes a stochastic programming problem. We can state the problem in such a way that one looks for a minimum lower bound η of the objective function under the constraint that the probability ψ of violation of that bound is prescribed ([83], [147])

$$\alpha^+ = \min \eta$$
$$\text{s.t.:} \begin{cases} \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}} \ge \eta \right) = \psi \\ \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} \\ \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 \end{cases} \tag{3.30}$$

The strength is distributed normally

For sake of simplicity of notation, we denote the plastic dissipation

$$\theta(\omega) = \sum_{k=1}^{m}\sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}r_i(\omega)\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \tag{3.31}$$

With this abbreviation the first constraint of (3.30) can be rewritten as:

$$\mathrm{Prob}(\theta \geq \eta) = 1 - \mathrm{Prob}(\theta \leq \eta) = 1 - \mathrm{Prob}\left(\frac{\theta - \mu_\theta}{\sigma_\theta} \leq \frac{\eta - \mu_\theta}{\sigma_\theta}\right) = \psi \tag{3.32}$$

In (3.32), $\mu_\theta, \sigma_\theta$ are mean value and standard deviation of $\theta(\omega)$. We can see in the inequality

$$\frac{\theta - \mu_\theta}{\sigma_\theta} \leq \frac{\eta - \mu_\theta}{\sigma_\theta} \tag{3.33}$$

that the left hand side is the normalized random variable with zero mean and unit variance. Hence the probabilistic condition (3.32) is replaced by

$$\psi = 1 - \Phi\left(\frac{\eta - \mu_\theta}{\sigma_\theta}\right) = \Phi\left(\frac{\mu_\theta - \eta}{\sigma_\theta}\right) \tag{3.34}$$

Setting $\psi = \Phi(\kappa)$ we have $\Phi^{-1}(\psi) = \kappa = \dfrac{\mu_\theta - \eta}{\sigma_\theta}$ or $\mu_\theta - \kappa\sigma_\theta = \eta$.

Therefore the separate chance constrained program (3.35) has the deterministic equivalent with $\kappa = \Phi^{-1}(\psi)$:

$$\alpha^+ = \min \eta$$
$$\text{s.t.:}\begin{cases} \mu_\theta - \kappa\sigma_\theta = \eta \\ \sum_{k=1}^{m}\dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i\dot{\mathbf{u}} = \mathbf{0} \\ \mathbf{D}_v\dot{\mathbf{e}}_{ik} = \mathbf{0} \\ \sum_{k=1}^{m}\sum_{i=1}^{N_e}\dot{\mathbf{e}}_{ik}^T\mathbf{t}_{ik} - 1 = 0 \end{cases} \tag{3.36}$$

The mean value of $\theta(\omega)$ is as follows:

$$\mu_{\theta(\omega)} = \sum_{k=1}^{m}\sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}\mu_i\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \tag{3.37}$$

where μ_i is the mean value of the strength $r_i(\omega)$.

The variance and standard deviation of $\theta(\omega)$ are computed as

$$\begin{aligned}
\mathrm{Var}\left[\theta(\omega)\right] &= \mathrm{Var}\left[\sum_{k=1}^{m}\sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}r_i(\omega)\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\right] \\
&= \mathrm{Var}\left[\sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}r_i(\omega)\sum_{k=1}^{m}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\right] \\
&= \left(\sum_{k=1}^{m}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\right)^2 \mathrm{Var}\left[\sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}r_i(\omega)\right]
\end{aligned} \tag{3.38}$$

From (3.57) we have:

$$\begin{aligned}
\sigma_{\theta(\omega)} = \sqrt{\mathrm{Var}\left[\theta(\omega)\right]} &= \sqrt{\mathrm{Var}\left[\sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}r_i(\omega)\sum_{k=1}^{m}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\right]} \\
&= \sqrt{\left(\sum_{k=1}^{m}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\right)^2 \mathrm{Var}\left[\sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}r_i(\omega)\right]} \\
&= \sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}\sigma_i\sum_{k=1}^{m}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2} \\
&= \sum_{i=1}^{N_e}\sum_{k=1}^{m}\sqrt{\frac{2}{3}}\sigma_i\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}
\end{aligned} \tag{3.39}$$

Finally, we can write the discretized upper bound of shakedown limit load moving the chance constraint to the objective function for the case of normally distributed strength:

$$\begin{aligned}
&\alpha^+ = \min \sum_{k=1}^{m}\sum_{i=1}^{Ne}\sqrt{\frac{2}{3}}\left(\mu_i - \kappa\sigma_i\right)\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2} \\
&\text{s.t.:}\ \begin{cases}
\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i\dot{\mathbf{u}} = \mathbf{0} \\
\mathbf{D}_v\dot{\mathbf{e}}_{ik} \quad = \mathbf{0} \qquad\qquad \forall i = \overline{1, Ne},\quad \forall k = \overline{1, m} \\
\sum_{k=1}^{m}\sum_{i=1}^{Ne}\dot{\mathbf{e}}_{ik}^T\mathbf{t}_{ik} - 1 = 0
\end{cases}
\end{aligned} \tag{3.40}$$

The strength is distributed lognormally

If the random strength $r_i(\omega)$ is distributed lognormally no closed form probability distribution exists for the sum

$$\theta(\omega) = \sum_{i=1}^{Ne}\sum_{k=1}^{m}\sqrt{\frac{2}{3}}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\cdot r_i(\omega) = \sum_{k=1}^{m}D_p(\omega)$$

Either an approximate probability distribution is derived mathematically or the assumption that a sum of independent lognormal random variables is also lognormally distributed is used and the sum is approximated by a single lognormal random variable [148].

The probability distribution of the plastic dissipation $D_p(\omega)$ in (3.30) and thus the transformation of (3.30) into an equivalent deterministic form can only be obtained as an approximation. Nevertheless, there is a duality between lower bound and upper bound formulation. Consequently, one can assume the equivalent deterministic of (3.30) as (3.41)

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{Ne} e^{\mu_i - \kappa_i \sigma_i} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall\, i = \overline{1, N_e} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} & \forall\, i = \overline{1, Ne}, \quad \forall\, k = \overline{1, m} \\ \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 \end{cases} \tag{3.41}$$

and then prove that the maximum problem (3.28) and the minimum problem (3.41) are dual to each other.

- **Duality approach to chance constrained programming**

In this section, we show that problem (3.41) is an equivalent deterministic program of the stochastic program (3.30) by investigating the duality property of (3.28) and (3.41). Starting from problem (3.41) as primal problem the minimum problem with restrictions is transformed into an unrestricted problem by the Lagrangian $L = L(\dot{\mathbf{e}}_{ik}, \dot{\mathbf{u}}, \alpha, \boldsymbol{\beta}_{ik}, \boldsymbol{\gamma}_{ik})$. With the Lagrange factors $\alpha, \boldsymbol{\beta}_{ik}, \boldsymbol{\gamma}_{ik}$ it holds

$$L = \sum_{i=1}^{Ne} \left\{ \sum_{k=1}^{m} \left(e^{\mu_i - \kappa\sigma_i} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}} \right) - \sum_{k=1}^{m} \boldsymbol{\gamma}_{ik}^T \mathbf{D}_v \dot{\mathbf{e}}_{ik} + \boldsymbol{\beta}_i^T \left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right) \right\} - \alpha \left(\sum_{i=1}^{Ne} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 \right) \tag{3.42}$$

The Lagrange dual function (3.42) can be rewritten as:

$$L = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\frac{e^{\mu_i - \kappa\sigma_i} \dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}}} - \boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \alpha\, \mathbf{t}_{ik} \right)^T \dot{\mathbf{e}}_{ik} + \sum_{i=1}^{Ne} \boldsymbol{\beta}_k^T \hat{\mathbf{B}}_k \dot{\mathbf{u}} + \alpha \tag{3.43}$$

Due to the existence of a dual solution having no duality gap the primal α^+, the function $\min_{\dot{\mathbf{e}}_{ik}, \dot{\mathbf{u}}} L(\dot{\mathbf{e}}_{ik}, \dot{\mathbf{u}}, \alpha, \boldsymbol{\beta}_i, \boldsymbol{\gamma}_{ik})$ must have a finite value α. Therefore,

$$\left(\frac{e^{\mu_i - \kappa\sigma_i} \dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}}} - \boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \alpha\, \mathbf{t}_{ik} \right)^T \dot{\mathbf{e}}_{ik} \geq 0 \tag{3.44}$$

must be satisfied in (3.43) because otherwise we always have $\min_{\dot{\mathbf{e}}_{ik}, \dot{\mathbf{u}}} L(\dot{\mathbf{e}}_{ik}, \dot{\mathbf{u}}, \alpha, \boldsymbol{\beta}_i, \boldsymbol{\gamma}_{ik}) \to -\infty$.

At the maximum the Lagrangian $L(\dot{\mathbf{e}}_{ik},\dot{\mathbf{u}},\alpha,\boldsymbol{\beta}_k,\boldsymbol{\gamma}_{ik})$ has a saddle point, so that the optimal value is the solution of

$$\max_{\alpha,\boldsymbol{\beta}_{ik},\boldsymbol{\gamma}_{ik}}\ \min_{\dot{\mathbf{e}}_{ik},\dot{\mathbf{u}}} L(\dot{\mathbf{e}}_{ik},\dot{\mathbf{u}},\alpha,\boldsymbol{\beta}_i,\boldsymbol{\gamma}_{ik}). \tag{3.45}$$

The necessary conditions for the minimum are

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\mathbf{e}}_{ik}} &= -\boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik} + e^{\mu_i-\kappa\sigma_i}\frac{\partial}{\partial \dot{\mathbf{e}}_{ik}}\frac{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}} \\
&= -\boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik} + e^{\mu_i-\kappa\sigma_i}\left(\frac{2\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}} - \frac{\left(\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}\right)^2\dot{\mathbf{e}}_{ik}}{\left(\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}\right)^3}\right) \\
&= -\boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik} + e^{\mu_i-\kappa\sigma_i}\left(\frac{\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}}\right) = \mathbf{0}
\end{aligned} \tag{3.46}$$

$$\frac{\partial L}{\partial \dot{\mathbf{u}}} = \boldsymbol{\beta}_i^T\hat{\mathbf{B}}_i = \mathbf{0} \tag{3.47}$$

Therefore, at the minimum

$$\left(\frac{e^{\mu_i-\kappa\sigma_i}\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}} - \boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik}\right)^T\dot{\mathbf{e}}_{ik} = 0 \tag{3.48}$$

With (3.43), (3.46) and (3.47) the dual objective function is

$$l(\alpha,\boldsymbol{\beta}_i,\boldsymbol{\gamma}_{ik}) = \min_{\dot{\mathbf{e}}_{ik},\dot{\mathbf{u}}} L(\dot{\mathbf{e}}_{ik},\dot{\mathbf{u}},\alpha,\boldsymbol{\beta}_i,\boldsymbol{\gamma}_{ik}) = \alpha \tag{3.49}$$

Furthermore, it is possible to prove that condition (3.44) at the minimum is also equivalent to following equality:

$$\left\|\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik}\right\| = e^{\mu_i-\kappa\sigma_i}\,. \tag{3.50}$$

Indeed, we can always choose a strain rate $\dot{\mathbf{e}}_{ik}$ parallel to the vector $\boldsymbol{\gamma}_{ik}+\boldsymbol{\beta}_k+\alpha\,\mathbf{t}_{ik}$ and therefore

$$\begin{aligned}
&\left(\frac{e^{\mu_i-\kappa\sigma_i}\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}} - \boldsymbol{\gamma}_{ik} - \boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik}\right)^T\dot{\mathbf{e}}_{ik} = \frac{e^{\mu_i-\kappa\sigma_i}\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}}} - \left(\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik}\right)^T\dot{\mathbf{e}}_{ik} = \\
&= e^{\mu_i-\kappa\sigma_i}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}} - \left\|\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_k + \alpha\,\mathbf{t}_{ik}\right\|\cdot\left\|\dot{\mathbf{e}}_{ik}\right\| = e^{\mu_i-\kappa\sigma_i}\left\|\dot{\mathbf{e}}_{ik}\right\| - \left\|\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_k + \alpha\,\mathbf{t}_{ik}\right\|\cdot\left\|\dot{\mathbf{e}}_{ik}\right\| = 0
\end{aligned}$$

from which equation (3.44) follows. Similarly we get

$$\left\|\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik}\right\| \le e^{\mu_i-\kappa\sigma_i} \tag{3.51}$$

Now we can state the dual problem of (3.41)

$$\alpha^{-} = \max_{\alpha,\boldsymbol{\beta}_i,\boldsymbol{\gamma}_{ik}} \alpha$$
$$\text{s.t.:} \quad \begin{cases} \left\| \boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \alpha\, \mathbf{t}_{ik} \right\| \le e^{\mu_i - \kappa\sigma_i} \\ \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \boldsymbol{\beta}_i = \mathbf{0} \end{cases} \tag{3.52}$$

Problem (3.52) can be written with the von Mises yield function as follows after some transformations; for more detail the reader can refer to the treatment of the deterministic problem in Vu [129]

$$\alpha^{-} = \max_{\alpha,\boldsymbol{\beta}_i,\boldsymbol{\gamma}_{ik}} \alpha$$
$$\text{s.t.:} \quad \begin{cases} f\left[\alpha\, \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right] \le e^{\mu_i - \kappa\sigma_i} \\ \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \boldsymbol{\beta}_i = \mathbf{0} \end{cases} \tag{3.53}$$

The form (3.53) is also exactly the discretized form of the lower bound shakedown limit, which is formulated by Melan's static theorem. The problem (3.53) is identical with problem (3.28). This means that (3.28) and (3.41) are dual to each other. Moreover, problem (3.41) is the equivalent deterministic form of (3.30).

In [149], The duality property of upper bound and lower bound in the case of normally distributed strength is presented.

The primal and dual problem can be written in a unified for normally distributed or lognormally distributed yield stress:

Primal problem:

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{Ne} \overline{Y} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}$$
$$\text{s.t.:} \quad \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} \\ \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 \end{cases} \tag{3.54}$$

Dual problem:

$$\alpha^{-} = \max \alpha$$
$$\text{s.t.:} \quad \begin{cases} \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{B}^T \overline{\boldsymbol{\rho}} = \mathbf{0} \\ f\left[\alpha\, \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right] \le \overline{Y} \end{cases} \tag{3.55}$$

in which for the normal distribution of strength

$$\overline{Y} = \sqrt{\frac{2}{3}}(\mu_i - \kappa\sigma_i) \tag{3.56}$$

and the lognormal distribution of strength

$$\overline{Y} = \sqrt{\frac{2}{3}} e^{\mu_i - \kappa\sigma_i} . \tag{3.57}$$

It is worth noting that in (3.56) μ_i, σ_i present mean value and standard deviation of strength while in (3.57) they are parameters of the lognormal distribution which are computed as (3.1)

Problems (3.54), (3.55) can be solved simultaneously by a dual algorithm which is presented in next section.

3.3.2 Structures under uncertain loading

The shakedown analysis problem under uncertain condition of loading is investigated in this section. The strength of material is considered deterministic in this situation. A new formulation will be established by the kinematic approach. We start with the upper bound deterministic problem (2.80) :

$$\alpha^{+} = \min \frac{\dot{W}_{in}}{\dot{W}_{ext}}$$

$$\text{with} \quad \dot{W}_{in} = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}$$

$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} \quad = \mathbf{0} \\ \dot{W}_{ext} = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} = 1 \end{cases}$$

By problem (2.80), the upper bound shakedown limit may be found in the case that the load and strength of the material are deterministic.

We consider a situation in which the loads, which are applied to the structure, are random variables. As a result, the stress vectors $\mathbf{t}_{ik}$ in (1) are random variables. We assume that they follow normal distributions with mean value $E(\mathbf{t}_{ik}) = \boldsymbol{\mu}_{ik}$ and variance $\mathrm{Var}(\mathbf{t}_{ik}) = \boldsymbol{\sigma}_{ik}^2$. The third constraint of (2.80) is not always satisfied because of the stochastic property of $\mathbf{t}_{ik}$, we must set it with a pre-described probability (or given reliability level ψ_p). In this case, problem (2.80) becomes a stochastic programming problem:

$$\alpha^{+} = \min \frac{\dot{W}_{in}}{\dot{W}_{ext}}$$

$$\text{with} \quad \dot{W}_{in} = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \tag{3.58}$$

$$\text{s.t.}: \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall k = \overline{1, m} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} & \forall i = \overline{1, Ne}, \quad \forall k = \overline{1, m} \\ \text{Prob}\left(\dot{W}_{ext} = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} < 1 \right) = \psi_p \end{cases}$$

Note that we seek the minimum over the $\dot{\mathbf{e}}_{ik}$ which satisfy the constraints. Thus $\dot{\mathbf{e}}_{ik}$ is not stochastic.

We denote some new variables

$$\begin{cases} \mu = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \mu_{ik} \\ \sigma^2 = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \sigma_{ik}^2 \\ \sum_{i=1}^{Ne} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik}(\omega) = \theta(\omega) \end{cases} \tag{3.59}$$

Let us compute the mean value or expectation and variance of power of the loading (external power) $\theta(\omega) = W_{ex}$.

- Mean value of $\theta(\omega)$

$$\begin{aligned} &\mathrm{E}(\mathbf{t}_{ik}) = \boldsymbol{\mu}_{ik} \\ &\mathrm{E}(\theta) = \mu_\theta = \mathrm{E}\left(\sum_{i=1}^{Ne} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik}(\omega) \right) = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} \end{aligned} \tag{3.60}$$

- Variance of $\theta(\omega)$:

$$\begin{aligned} &\mathrm{Var}(\mathbf{t}_{ik}) = \boldsymbol{\sigma}_{ik}^2 \\ &\mathrm{Var}(\theta) = \sigma_\theta^2 = \mathrm{Var}\left(\sum_{i=1}^{Ne} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik}(\omega) \right) \\ &\mathrm{Var}(\theta) = \sigma_\theta^2 = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} \end{aligned} \tag{3.61}$$

in which $\mathbf{V}_{ik}$ is the covariance matrix of vector $\mathbf{t}_{ik}(\omega)$.

Consider probability constraints and let us make some transformations:

$$\begin{aligned}
&\mathrm{Prob}(\theta - 1 < 0) = \psi \\
&\mathrm{Prob}\left(\frac{\theta - \mu_\theta}{\sigma_\theta} < \frac{1 - \mu_\theta}{\sigma_\theta} \right) = \Phi\left(\frac{1 - \mu_\theta}{\sigma_\theta} \right) = \psi \\
&\frac{1 - \mu_\theta}{\sigma_\theta} = \Phi^{-1}(\psi) = \kappa \\
&1 = \mu_\theta + \kappa \sigma_\theta
\end{aligned} \tag{3.62}$$

By substituting (3.60), (3.61) into (3.62) we arrive at the mean value $\mathrm{E}(W_{ex})$ of the power of the loading:

$$W_{ex} = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) = 1 \tag{3.63}$$

Finally we have the equivalent deterministic formulation for the upper bound problem:

$$\begin{aligned}
&\min \frac{\dot{W}_{in}}{\dot{W}_{ex}} \left(= \alpha^+ \right) \\
&\text{with} \quad \dot{W}_{in} = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \\
&\text{s.t.:} \begin{cases}
\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall i = \overline{1, N_e} \\
\mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} & \forall i = \overline{1, N_e}, \quad \forall k = \overline{1, m} \\
\dot{W}_{ex} = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) = 1
\end{cases}
\end{aligned} \tag{3.64}$$

3.3.3 Structures under uncertain strength and loading

In case that the strength and loading are random variables, the formulation is deduced directly from arguments presented in sections (3.2.1, 3.2.2):

$$\begin{aligned}
&\alpha^+ = \min \eta \\
&\text{s.t.:} \begin{cases}
\mathrm{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}} \geq \eta \right) = \psi_r \\
\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} \\
\mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} \\
\mathrm{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} < 1 \right) = \psi_{\mathrm{p}}
\end{cases}
\end{aligned} \tag{3.65}$$

in which ψ_r, ψ_p are reliability levels corresponding to random strength and random load, respectively.

Using normal distribution for both strength and loads, problem (3.40) and (3.64) are combined to derive the upper bound equivalent deterministic programming problem:

$$\begin{aligned}
&\min \frac{\dot{W}_{in}}{\dot{W}_{ex}} \left(=\alpha^{+}\right) \\
&\text{with} \quad \dot{W}_{in} = \sum_{k=1}^{m}\sum_{i=1}^{Ne}\sqrt{\frac{2}{3}}\left(r_i - \kappa_r \sigma_i\right)\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \\
&\text{s.t.}: \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall i = \overline{1, N_e} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} & \forall i = \overline{1, N_e}, \quad \forall k = \overline{1, m} \\ W_{ex} = \sum_{i=1}^{Ne}\sum_{k=1}^{m}\left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}}\right) = 1 \end{cases}
\end{aligned} \tag{3.66}$$

Similar, if the strength is distributed lognormally and the loads are distributed normally, the equivalent deterministic upper bound formulation is as follows:

$$\begin{aligned}
&\min \frac{\dot{W}_{in}}{\dot{W}_{ex}} \left(=\alpha^{+}\right) \\
&\text{with} \quad \dot{W}_{in} = \sum_{k=1}^{m}\sum_{i=1}^{Ne}\sqrt{\frac{2}{3}} e^{\left(r_i - \kappa_r \sigma_i\right)}\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \\
&\text{s.t.}: \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall i = \overline{1, N_e} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} & \forall i = \overline{1, N_e}, \quad \forall k = \overline{1, m} \\ W_{ex} = \sum_{i=1}^{Ne}\sum_{k=1}^{m}\left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}}\right) = 1 \end{cases}
\end{aligned} \tag{3.67}$$

where κ_r, κ_p are the inverse CDF corresponding to random strength and random load, respectively.

3.3.4 Algorithms for stochastic shakedown programming

- **Kinematic algorithm - A1**

This new algorithm is employed to solve nonlinear programming problems (3.66) and (3.67).

Algorithm A1 allows to compute kinematic shakedown load factor in following situations :

- Deterministic load, deterministic strength
- Deterministic load, normally distributed strength
- Deterministic load, lognormally distributed strength
- Normally distributed load, deterministic strength

For convenience, we use notation as in (3.56, 3.57) :

for the normal distribution of strength

$$\overline{Y} = \sqrt{\frac{2}{3}}\left(\mu_i - \kappa\sigma_i\right)$$

and the lognormal distribution of strength

$$\overline{Y} = \sqrt{\frac{2}{3}} e^{\mu_i - \kappa\sigma_i} .$$

A penalty method is used to satisfy the compatibility condition. To this end, let us write the penalty function as

$$F_p = \sum_{i=1}^{Ne} \left[\begin{array}{l} \sum_{k=1}^{m} \overline{Y}_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} + \ldots \\ + \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^T \mathbf{D}_v \dot{\mathbf{e}}_{ik}\right) + \frac{c}{2}\left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}}\right)^T \left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}}\right) \end{array} \right] \qquad (3.68)$$

Where c is a penalty parameter, $c \gg 1$. The parameter c may be dependent on integration points and c should be adjusted to fit different compatibility condition. However, for the shake of simplicity, c is let to be constant everywhere at this stage.

Following (3.68), problem (3.66) becomes

$$\begin{aligned} & \alpha^+ = \min F_p \\ & \text{s.t.:} \quad \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) = 1 \end{aligned} \qquad (3.69)$$

Using Lagrange multipliers method, we convert (3.69) into an unconstrained programming problem with the corresponding Lagrange function

$$L = F_p - \lambda \left[\sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) - 1 \right] \qquad (3.70)$$

with λ is Lagrange multiplier.

Karush–Kuhn–Tucker optimality conditions (KKT optimality conditions) states that the conditions for a minimum local solution of (3.70) is

$$\begin{cases} \dfrac{\partial L}{\partial \dot{\mathbf{e}}_{ik}} = \left(\overline{Y}_i \dfrac{\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2}} \right) + c\mathbf{D}_M \dot{\mathbf{e}}_{ik} + c\left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} \right) \\ \qquad\qquad + \alpha \left(\boldsymbol{\mu}_{ik} + \kappa \dfrac{\mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon^2}} \right) = 0 & (a) \\ \dfrac{\partial L}{\partial \dot{\mathbf{u}}} = -c\mathbf{B}_i^T \left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} \right) = 0 & (b) \\ \dfrac{\partial L}{\partial \lambda} = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) - 1 = 0 & (c) \end{cases} \tag{3.71}$$

For convenience, let us rewrite (3.71) as

$$\begin{cases} \mathbf{f}_{ik} = \overline{Y}_i \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon^2}\, \dot{\mathbf{e}}_{ik} \\ \quad + c\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2} \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon^2}\, \mathbf{D}_M \dot{\mathbf{e}}_{ik} \\ \quad + c\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2} \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon^2} \left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} \right) \\ \quad - \lambda \left(\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2} \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon^2}\, \boldsymbol{\mu}_{ik} + \kappa \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2}\, \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} \right) = \mathbf{0} & (a) \\ \mathbf{h}_{ik} = c\mathbf{B}_i^T \left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} \right) \qquad = \mathbf{0} & (b) \\ \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) - 1 \qquad = 0 & (c) \end{cases} \tag{3.72}$$

We apply Newton's method for solving the (3.72) system of nonlinear equations:

$$\begin{cases} \mathbf{H}_{ik} d\dot{\mathbf{e}}_{ik} + c\sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \left(\sum_{k=1}^{m} d\dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i d\dot{\mathbf{u}} \right) \\ \qquad - \left(\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2} \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon^2}\, \boldsymbol{\mu}_{ik} + \kappa \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2}\, \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} \right) d\lambda = -\mathbf{f}_{ik} & (a) \\ c\hat{\mathbf{B}}_i \left(\sum_{k=1}^{m} d\dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i d\dot{\mathbf{u}} \right) = -\mathbf{h}_i & (b) \\ \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\boldsymbol{\mu}_{ik}^T + \kappa \dfrac{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}}} \right) d\dot{\mathbf{e}}_{ik} = 1 - \sum_{k=1}^{Ne} \sum_{i=1}^{m} \left(\dot{\mathbf{e}}_{ik}^T \boldsymbol{\mu}_{ik} + \kappa \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) & (c) \end{cases} \tag{3.73}$$

In the system (3.73) $d\dot{\mathbf{e}}_{ik}, d\dot{\mathbf{u}}$ denote the incremental vectors of strain rate and displacement respectively, $d\lambda$ denotes the incremental value of λ. In order to ensure the convergence of algorithm, instead of the exact form of matrix $\mathbf{H}_{ik}$ one can use its approximation as (3.74)

$$\mathbf{H}_{ik} \approx \left(\overline{Y}_i \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon^2}\right) \mathbf{I}_{ik} + c\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2}\sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon^2}\,\mathbf{D}_M - \lambda\kappa\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon^2}\,\mathbf{V}_{ik} \tag{3.74}$$

After some mathematical transformations we get the following system

$$\mathbf{S}\,d\dot{\mathbf{u}} = -\mathbf{S}\dot{\mathbf{u}} + (\lambda + d\lambda)\mathbf{f}_{u2} \tag{3.75}$$

where

$$\begin{cases} \mathbf{S} = \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \mathbf{E}_i^{-1} \hat{\mathbf{B}}_i \\ \mathbf{f}_{u2} = \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i \mathbf{E}_i^{-1} \sum_{k=1}^{m} \mathbf{H}_{ik}^{-1} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\,\boldsymbol{\mu}_{ik} \end{cases} \tag{3.76}$$

and

$$\mathbf{E}_i = \left(\frac{\mathbf{I}_i}{c} + \sum_{k=1}^{m} \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\,\mathbf{H}_{ik}^{-1}\right) \tag{3.77}$$

The system (3.75) with the second terms on right side may be interpreted as the linear system arising in purely elastic computations with the global stiffness matrix $\mathbf{S}$. The matrix $\mathbf{E}_i^{-1}$ plays the role of the elastic matrix at the smoothing domain containing edge i while the vector

$$\mathbf{f} = \mathbf{f}_{u2} = \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i \mathbf{E}_i^{-1} \sum_{k=1}^{m} \mathbf{H}_{ik}^{-1} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\,\boldsymbol{\mu}_{ik} \tag{3.78}$$

is considered as the vector of nodal forces applied on the structure.

Solving system (3.73) and after some transformations we have the incremental vectors of displacement and strain rate as

$$\begin{cases} d\dot{\mathbf{u}} = d\dot{\mathbf{u}}_1 + (\lambda + d\lambda)\,d\dot{\mathbf{u}}_2 & \text{(a)} \\ d\dot{\mathbf{e}}_{ik} = (d\dot{\mathbf{e}}_{ik})_1 + (\lambda + d\lambda)(d\dot{\mathbf{e}}_{ik})_2 & \text{(b)} \end{cases} \tag{3.79}$$

where

$$\begin{cases} d\dot{\mathbf{u}}_1 = -\dot{\mathbf{u}} \\ d\dot{\mathbf{u}}_2 = \mathbf{S}^{-1}\mathbf{f}_{u2} \\ (d\dot{\mathbf{e}}_{ik})_1 = -\dot{\mathbf{e}}_{ik} \\ (d\dot{\mathbf{e}}_{ik})_2 = \left[\mathbf{H}_{ik}^{-1} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\,\boldsymbol{\mu}_{ik}\right] \\ \qquad + \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\,\mathbf{H}_{ik}^{-1} \mathbf{E}_i^{-1} \hat{\mathbf{B}}_i\, d\dot{\mathbf{u}}_2 \\ \qquad - \sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\,\mathbf{H}_{ik}^{-1} \mathbf{E}_i^{-1} \sum_{k=1}^{m} \mathbf{H}_{ik}^{-1} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\sqrt{\dot{\mathbf{e}}_{ik}^T \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\,\boldsymbol{\mu}_{ik} \end{cases} \tag{3.80}$$

The vectors $d\dot{\mathbf{u}}, d\dot{\mathbf{e}}_{ik}$ are Newton directions which guarantee that a step along them leads to a decrease of the objective function. The incremental value of λ is computed by substituting (3.65b) into (3.59c):

$$(\lambda + d\lambda) = \frac{1 - \sum_{k=1}^{N_e}\sum_{i=1}^{m}\left(\dot{\mathbf{e}}_{ik}^T\boldsymbol{\mu}_{ik} + \kappa\sqrt{\dot{\mathbf{e}}_{ik}^T\mathbf{V}_{ik}\dot{\mathbf{e}}_{ik} + \varepsilon_0^2}\right) - \sum_{i=1}^{Ne}\sum_{k=1}^{m}\left(\boldsymbol{\mu}_{ik}^T + \kappa\frac{\dot{\mathbf{e}}_{ik}^T\mathbf{V}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\mathbf{V}_{ik}\dot{\mathbf{e}}_{ik} + \varepsilon_0^2}}\right)(d\mathbf{e}_{ik})_1}{\sum_{i=1}^{Ne}\sum_{k=1}^{m}\left(\boldsymbol{\mu}_{ik}^T + \kappa\frac{\dot{\mathbf{e}}_{ik}^T\mathbf{V}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\mathbf{V}_{ik}\dot{\mathbf{e}}_{ik} + \varepsilon_0^2}}\right)(d\mathbf{e}_{ik})_2} \tag{3.81}$$

Based on (3.79) we can update the displacement vector and strain rate vector. The new vectors $\dot{\mathbf{u}}$ and $\dot{\mathbf{e}}_{ik}$ tend to satisfy equations (3.86b),(3.87a) simultaneously. By forcing them to fulfil (3.86) we get Lagrange multiplier updated as in (3.96). Iterating these steps may drive us to a stable set of $\dot{\mathbf{u}}, \dot{\mathbf{e}}, \lambda$ satisfying all conditions (3.86b), (3.86c) and (3.87a). Details of the iterative algorithm are presented hereafter.

<u>*Algorithm A1:*</u>

• <u>*Step 1:*</u> Initialize displacement and strain rate vectors: $\dot{\mathbf{u}}^0$ and $\dot{\mathbf{e}}^0$ such that the normalized condition in (3.71) is satisfied

$$\dot{W}_{ex} = \sum_{k=1}^{N_e}\sum_{i=1}^{m}\left(\dot{\mathbf{e}}_{ik}^0\boldsymbol{\mu}_{ik} + \kappa\sqrt{\dot{\mathbf{e}}_{ik}^0\mathbf{V}_{ik}\dot{\mathbf{e}}_{ik}^0 + \varepsilon^2}\right) = 1\,.$$

Normally the fictitious elastic solution must be computed first in order to define the load domain Ω in terms of the fictitious elastic generalized stress $\hat{\boldsymbol{\sigma}}_{ik}^E$. Hence $\dot{\mathbf{u}}^0$ and $\dot{\mathbf{e}}^0$ may assume fictitious values (after being normalized) for their initialization. Set up initial values for the penalty parameter c and for ε_0. Set up convergence criteria, maximum number of iterations.

• <u>*Step 2:*</u> Calculate $\mathbf{S}$, $\mathbf{f}_{u2}$ from (3.76) at current values of $\dot{\mathbf{u}}$ and $\dot{\mathbf{e}}$

• <u>*Step 3:*</u> Calculate $(\lambda + d\lambda)$, $d\dot{\mathbf{u}}$, $d\dot{\mathbf{e}}_{ik}$ from equations (3.80), (3.79).

• <u>*Step 4:*</u> Calculate step size β_k by solving the sub-problem:

$$F_p\left(\dot{\mathbf{u}} + \beta_k d\dot{\mathbf{u}},\ \dot{\mathbf{e}} + \beta_k d\dot{\mathbf{e}}\right) \to \min$$

Update displacement, strain rate and λ as

$$\begin{aligned} \dot{\mathbf{u}} &= \dot{\mathbf{u}} + \beta_k d\dot{\mathbf{u}} && \text{(a)}\\ \dot{\mathbf{e}}_{ik} &= \dot{\mathbf{e}}_{ik} + \beta_k d\dot{\mathbf{e}}_{ik} && \text{(b)}\\ \lambda &= \lambda + d\lambda && \text{(c)} \end{aligned}$$

• <u>*Step 5:*</u> Check convergence criteria: if they are all satisfied go to step 6 otherwise repeat steps 2, 3 and 4.

• Step 6: Stop

The algorithm can fail due to some reasons: failure in computing the inverse matrix $\mathbf{s}^{-1}$ or unsuccessful initialization step, which after some iterations results in an unexpected form

$$\sum_{i=1}^{Ne}\sum_{k=1}^{m}\left(\dot{\mathbf{e}}_{ik}^{0}\boldsymbol{\mu}_{ik}+\kappa\sqrt{\dot{\mathbf{e}}_{ik}^{0}\mathbf{V}_{ik}\dot{\mathbf{e}}_{ik}^{0}+\varepsilon^{2}}\right)=0$$

or the maximum number of iterations is insufficient to get a convergent solution. If those obstacles do not exist, a solution set $\left(\bar{\dot{\mathbf{u}}},\bar{\dot{\mathbf{e}}},\bar{\alpha}\right)$ can be found.

- **Primal- Dual Algorithm A2**

The dual algorithm is proposed to solve simultaneously problems(3.54) and (3.55).

Primal problem (3.54):

$$\alpha^{+}=\min\sum_{k=1}^{m}\sum_{i=1}^{Ne}\bar{Y}\sqrt{\dot{\mathbf{e}}_{ik}^{T}\dot{\mathbf{e}}_{ik}+\varepsilon_{0}^{2}}$$

$$\text{s.t.:}\begin{cases}\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik}-\hat{\mathbf{B}}_{i}\dot{\mathbf{u}}=\mathbf{0}\\ \mathbf{D}_{v}\dot{\mathbf{e}}_{ik}\quad=\mathbf{0}\\ \sum_{k=1}^{m}\sum_{i=1}^{Ne}\dot{\mathbf{e}}_{ik}^{T}\mathbf{t}_{ik}-1=0\end{cases}$$

Dual problem (3.55) :

$$\alpha^{-}=\max\alpha$$

$$\text{s.t.:}\begin{cases}\sum_{i=1}^{Ne}\hat{\mathbf{B}}_{i}^{T}\bar{\boldsymbol{\rho}}_{i}=\mathbf{B}^{T}\bar{\boldsymbol{\rho}}=\mathbf{0}\\ f\left[\alpha\boldsymbol{\sigma}_{ik}^{E}+\bar{\boldsymbol{\rho}}_{i}\right]\leq\bar{Y}\end{cases}$$

The dual relationship between them was investigated in section 3.3.1.

Similar to Algorithm A1, we also used Penalty and Lagrange multiplier method to transform problem (3.54) in to unconstrained programming.

The Penalty function is as follows

$$F_{P}=\sum_{i=1}^{Ne}\left[\sum_{k=1}^{m}\bar{Y}\sqrt{\dot{\mathbf{e}}_{ik}^{T}\dot{\mathbf{e}}_{ik}+\varepsilon_{0}^{2}}+\frac{c}{2}\left(\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik}-\hat{\mathbf{B}}_{i}\dot{\mathbf{u}}\right)^{T}\left(\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik}-\hat{\mathbf{B}}_{i}\dot{\mathbf{u}}\right)+\frac{c}{2}\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik}^{T}\mathbf{D}_{\upsilon}\dot{\mathbf{e}}_{ik}\right]\quad(3.82)$$

Lagrange function:

$$L=F_{p}-\lambda\left[\sum_{i=1}^{Ne}\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik}^{T}\mathbf{t}_{ik}-1\right]\quad(3.83)$$

The KKT condition for optimal solution of (3.83) :

$$\begin{cases} \dfrac{\partial L}{\partial \dot{\mathbf{e}}_{ik}} = \left(\overline{Y} \dfrac{\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}} \right) + c\mathbf{D}_\upsilon \dot{\mathbf{e}}_{ik} + c\left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} \right) + \alpha \mathbf{t}_{ik} = 0 & \text{(a)} \\ \dfrac{\partial L}{\partial \dot{\mathbf{u}}} = -c\mathbf{B}_i^T \left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} \right) = 0 & \text{(b)} \\ \dfrac{\partial L}{\partial \alpha} = \left(\sum_{i=1}^{Ne} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 \right) = 0 & \text{(c)} \end{cases} \tag{3.84}$$

We define new variables:

$$\boldsymbol{\beta}_i = -c\left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} \right) \tag{3.85}$$

$$\boldsymbol{\gamma}_{ik} = -c\mathbf{D}_\upsilon \dot{\mathbf{e}}_{ik} \tag{3.86}$$

the stationarity conditions (3.84) combined with (3.85), (3.86) become:

$$\begin{cases} \left(\overline{Y} \dfrac{\dot{\mathbf{e}}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}} \right) + \boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \alpha \mathbf{t}_{ik} = \mathbf{0} \\ \sum_{i=1}^{Ne} \left[-\mathbf{B}_i^T \boldsymbol{\beta}_i \right] = \mathbf{0} \\ \left(\sum_{i=1}^{Ne} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 \right) = 0 \\ \boldsymbol{\gamma}_{ik} = c\mathbf{D}_\upsilon \dot{\mathbf{e}}_{ik} \\ \boldsymbol{\beta}_i = c\left(\sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \mathbf{B}_i \dot{\mathbf{u}} \right) \end{cases} \tag{3.87}$$

Using Newton's method to solve the system of nonlinear equations (3.87), after transformations we obtain $d\dot{\mathbf{u}}$, $d\dot{\mathbf{e}}_{ik}$, $d\alpha$, $d\boldsymbol{\beta}_i$, $d\boldsymbol{\gamma}_{ik}$:

$$d\dot{\mathbf{u}} = d\dot{\mathbf{u}}_1 + d\dot{\mathbf{u}}_2 (\alpha + d\alpha) \tag{3.88}$$

$$d\dot{\mathbf{e}}_{ik} = (d\dot{\mathbf{e}}_{ik})_1 + (d\dot{\mathbf{e}}_{ik})_2 (\alpha + d\alpha) \tag{3.89}$$

$$d\boldsymbol{\gamma}_{ik} = c\mathbf{D}_\upsilon d\dot{\mathbf{e}}_{ik} - (\boldsymbol{\gamma}_{ik} - c\mathbf{D}_\upsilon \dot{\mathbf{e}}_{ik}) \tag{3.90}$$

$$d\boldsymbol{\beta}_i = (d\boldsymbol{\beta}_i)_1 + (d\boldsymbol{\beta}_i)_2 (\alpha + d\alpha) \tag{3.91}$$

$$(\alpha + d\alpha) = \left[\frac{1 - \sum_{i=1}^{Ne} \sum_{k=1}^{m} \mathbf{t}_{ik}^T \left\{ \dot{\mathbf{e}}_{ik} + (d\dot{\mathbf{e}}_{ik})_1 \right\}}{\sum_{i=1}^{Ne} \sum_{k=1}^{m} \mathbf{t}_{ik}^T (d\dot{\mathbf{e}}_{ik})_2} \right] \tag{3.92}$$

$$
\begin{aligned}
d\dot{\mathbf{u}}_1 &= -\dot{\mathbf{u}} + \mathbf{S}^{-1}\sum_{i=1}^{Ne}\mathbf{B}_i^T\mathbf{E}_i^{-1}\sum_{k=1}^{m}\overline{\mathbf{M}}_{ik}^{-1}\left(\boldsymbol{\gamma}_{ik}+\boldsymbol{\beta}_i+\alpha\,\mathbf{t}_{ik}\right)\frac{\dot{\mathbf{e}}_{ik}^T\mathbf{e}_{ik}}{\sqrt{\dot{\mathbf{e}}_{ik}^T\mathbf{e}_{ik}+\varepsilon_0^2}}\\
d\dot{\mathbf{u}}_2 &= -\mathbf{S}^{-1}\sum_{i}^{NG}\mathbf{B}_i^T\mathbf{E}_i^{-1}\sum_{k}^{M}\overline{\mathbf{M}}_{ik}^{-1}\sqrt{\dot{\mathbf{e}}_{ik}^T\mathbf{e}_{ik}+\varepsilon_0^2}\,\mathbf{t}_{ik}\\
\mathbf{S} &= \sum_{i=1}^{Ne}\mathbf{B}_i^T\mathbf{E}_i^{-1}\mathbf{B}_i
\end{aligned}
\tag{3.93}
$$

$$
\begin{aligned}
\left(d\dot{\mathbf{e}}_{ik}\right)_1 &= +\overline{\mathbf{M}}_{ik}^{-1}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\left(-c\mathbf{D}_{\upsilon}\dot{\mathbf{e}}_{ik}-\left(d\boldsymbol{\beta}_i\right)_1\right)-\overline{\mathbf{M}}_{ik}^{-1}\left(\sqrt{2}k_{\nu}\dot{\mathbf{e}}_{ik}+\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\,\boldsymbol{\beta}_i\right)\\
\left(d\dot{\mathbf{e}}_{ik}\right)_2 &= -\overline{\mathbf{M}}_{ik}^{-1}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\left(d\boldsymbol{\beta}_i\right)_2-\overline{\mathbf{M}}_{ik}^{-1}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\,\mathbf{t}_{ik}
\end{aligned}
\tag{3.94}
$$

$$
\begin{aligned}
\left(d\boldsymbol{\beta}_i\right)_1 &= -\mathbf{E}_i^{-1}\sum_{k=1}^{m}\overline{\mathbf{M}}_{ik}^{-1}\left(\overline{Y}\,\mathbf{I}+c\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\,\mathbf{D}_{\upsilon}\right)\dot{\mathbf{e}}_{ik}\\
&\quad-\mathbf{E}_i^{-1}\left[\mathbf{B}_i\,d\dot{\mathbf{u}}_1-\left(\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik}-\mathbf{B}_i\dot{\mathbf{u}}\right)\right]-\boldsymbol{\beta}_i\\
\left(d\boldsymbol{\beta}_i\right)_2 &= -\mathbf{E}_i^{-1}\mathbf{B}_i\,d\dot{\mathbf{u}}_2-\mathbf{E}_i^{-1}\sum_{k=1}^{m}\overline{\mathbf{M}}_{ik}^{-1}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\,\mathbf{t}_{ik}
\end{aligned}
\tag{3.95}
$$

$$
\mathbf{E}_i = \frac{\mathbf{I}_{ik}}{c}+\sum_{k=1}^{m}\overline{\mathbf{M}}_{ik}^{-1}\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}
\tag{3.96}
$$

$$
\begin{aligned}
\mathbf{M}_{ik} &= \mathbf{I}_{ik}+\left(\boldsymbol{\gamma}_{ik}+\boldsymbol{\beta}_i+\alpha\,\mathbf{t}_{ik}\right)\frac{\dot{\mathbf{e}}_{ik}^T}{\sqrt{\dot{\mathbf{e}}_{ik}^T\mathbf{e}_{ik}+\varepsilon_0^2}}\\
\overline{\mathbf{M}}_{ik} &= \mathbf{M}_{ik}+c\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\,\mathbf{D}_{\upsilon}
\end{aligned}
\tag{3.97}
$$

It is worth noting that $d\dot{\mathbf{u}}, d\dot{\mathbf{e}}_{ik}, d\alpha$ are Newton directions and when moving along them we expect that the value of the objective function will reduce.

Algorithm A.2

Step 1. Initialize the displacement and strain rate vectors $\dot{\mathbf{u}}^0, \dot{\mathbf{e}}^0$ such that the normalized condition (3.84c) is satisfied:

$$
\sum_{i=1}^{Ne}\sum_{k=1}^{m}\mathbf{t}_{ik}^T\dot{\mathbf{e}}_{ik}^0 = 1
$$

Set all vectors equal to zero vectors:

$$
\begin{cases}\boldsymbol{\beta}_i^0 = \mathbf{0}\\ \boldsymbol{\gamma}_{ik}^0 = \mathbf{0}\end{cases}\qquad \forall\, i = \overline{1, Ne},\quad k = \overline{1, m}
$$

Set up initial values for penalty parameter c and for ε_0. Set up convergence criteria, maximum number of iterations allowed to carry out.

Step 2. Calculate $(d\dot{\mathbf{e}}_{ik})_1, (d\dot{\mathbf{e}}_{ik})_2$ at the current value of $\dot{\mathbf{e}}_{ik}$ from(3.94), $(\alpha + d\alpha)$ from(3.92), calculate $d\dot{\mathbf{e}}_{ik}$ from (3.89)

Step 3. Calculate $d\dot{\mathbf{u}}_1, d\dot{\mathbf{u}}_2$ from (3.93) and then calculate $d\dot{\mathbf{u}}$ from (3.88)

Step 4. Calculate a suitable *step-length* λ_k by solving the sub-problem

$$\min F_p\left(\dot{\mathbf{u}} + \lambda_k d\dot{\mathbf{u}}, \dot{\mathbf{e}} + \lambda_k d\dot{\mathbf{e}}\right)$$

Update displacement and strain rate as:

$$\begin{aligned}\dot{\mathbf{u}} &= \dot{\mathbf{u}} + \lambda_k d\dot{\mathbf{u}} \\ \dot{\mathbf{e}} &= \dot{\mathbf{e}} + \lambda_k d\dot{\mathbf{e}}\end{aligned}$$

Step 5. Calculate $d\boldsymbol{\beta}_i, d\boldsymbol{\gamma}_{ik}$ from (3.90), (3.91). Find a *step-length* λ_s such that:

$$\begin{aligned}&\lambda_s = \max \lambda \\ &\text{s.t.:} \quad \left\|\boldsymbol{\gamma}_{ik} + \boldsymbol{\beta}_i + \alpha \mathbf{t}_{ik} + \lambda(d\boldsymbol{\gamma}_{ik} + d\boldsymbol{\beta}_i + \mathbf{t}_{ik} d\alpha)\right\| \le \overline{Y}\end{aligned}$$

Update $\alpha, \boldsymbol{\beta}_i, \boldsymbol{\gamma}_{ik}$:

$$\begin{aligned}\alpha &= \alpha + \lambda_s d\alpha \\ \boldsymbol{\beta}_i &= \boldsymbol{\beta}_i + \lambda_s d\boldsymbol{\beta}_i \\ \boldsymbol{\gamma}_{ik} &= \boldsymbol{\gamma}_{ik} + \lambda_s d\boldsymbol{\gamma}_{ik}\end{aligned}$$

Step 6. Check the convergence criteria: if they are satisfied then stop the algorithm, otherwise repeat steps 2-5.

Algorithm A2 allows to compute upper and lower bound shakedown load factors in following situations :

- Deterministic load, deterministic strength
- Deterministic load, normally distributed strength
- Deterministic load, lognormally distributed strength

3.4 Convexity of the chance contrained programs

The constraints in the deterministic lower and upper bound are linear and therefore clearly convex and smooth (diferentiable). The upper and lower bound limit and shakedown problem are convex programs whose local maximum and minimum are the globel solution so that they are strict lower and upper bounds wich are true solutions by duality. Only under the approximation by a FEM dicretization this condition may not hold strictly.

Consider e.g. the von Mises yield condition

$$f\left[\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right]\le r_i \qquad \text{with} \qquad r_i=\sqrt{\frac{2}{3}}\sigma_y$$

which is linear in the yield stress σ_y. The probabilistic condition with the distribution function $F(r(\omega))$

$$\mathrm{Prob}\left[f\left(\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right)\le r_i(\omega)\right]\ge\psi_i$$

is not always convex because the distribution function is typically neither convex nor concave. Here we assume that the stress has a continuous distribution so that we have only to discuss the loss of convexity.

Let $F(r_i(\omega))$ be the distribution function of $r_i(\omega)$ then $F^{-1}(p)$ is the quantile function and it holds

$$\mathrm{Prob}\left[f\left(\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right)\le r_i(\omega)\right]=1-F\left(f\left(\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right)\right)\ge\psi_i$$

$$\mathrm{Prob}\left[f\left(\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right)\ge r_i(\omega)\right]=F\left(f\left(\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right)\right)\qquad\le\psi_i$$

with $\kappa_i=F^{-1}(\psi_i)$ or $\psi_i=F(\kappa_i)$ so that the deterministic equivalent constraint is

$$f\left(\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right)\le F^{-1}(\psi_i)=\kappa_i\,.$$

Therefore $F^{-1}(\psi_i)$ decides if the constraint is convex or concave. A rather detailed presentation of the currently known conditions for convexity and reference to original work is found in (Shapiro et al., 2014),[150].

Chapter 4

Numerical Applications

4.1 Two span continuous beam

We first consider the two span continuous beam with rectangular cross-section. The beam is subjected to two point forces as shown in figure 4.1. This test is investigated analytically by Sikorski and Borkowski in [64] for the deterministic problem and for normal distributions. The lognormal distributions have not been considered before. Let us determine the limit load factor in four situations:

+ Both strength and loading are deterministic

+ Loads are deterministic, strength is a random variable with the mean values $M_{0,1} = 2.0\ \mathrm{kNm}$, $M_{0,2} = 3.0\ \mathrm{kNm}$ corresponding to the first and the second span. The standard deviation is $\sigma_i = 0.1 M_{0,i}$ for each span.

+ Loads are random variables with the mean values $\overline{P_1} = \alpha\, 3\ \mathrm{kN}$, $\overline{P_2} = \alpha\, 2\ \mathrm{kN}$ and the standard deviations $\sigma_i = 0.05\overline{P_i}$. Each span has a deterministic yield moment with values $M_{0,1} = 2.0\ \mathrm{kNm}$, $M_{0,2} = 3.0\ \mathrm{kNm}$.

+ Loads and strength are random.

The given partial reliability levels are $\psi_s = \psi_p = 0.9999$ so that $\kappa_r = \Phi^{-1}(\psi_r) = \kappa_p = \Phi^{-1}(\psi_p) = \Phi^{-1}(0.9999) = 3.719$.

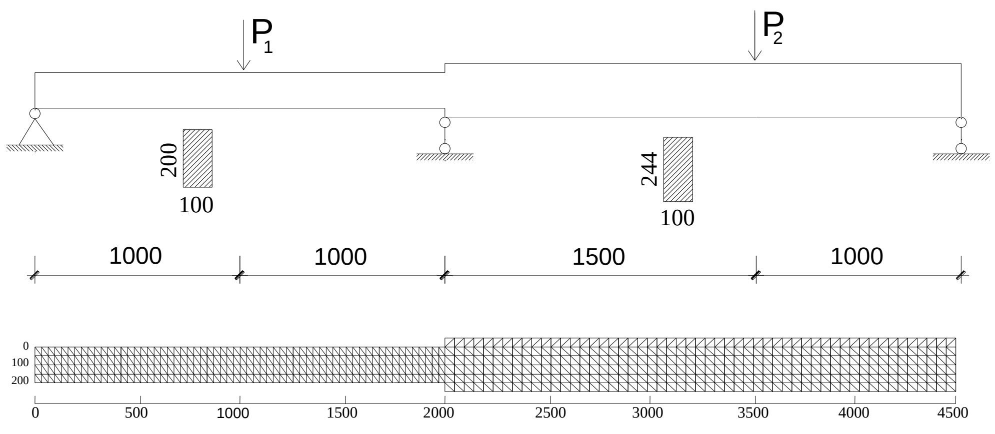


Figure 4. 1 Two-span beam and FE mesh with T3 elements

- *Random strength, loads are deterministic:*

The figure 4.1 shows the two-span beam with its FE mesh. The beam is modelled by 1350 T3 elements. In figure 4.2 the convergence of the limit load factors is shown for some cases of random strength. The convergent numerical solutions are 2.19, 1.51, 1.38 for deterministic, lognormal distribution and normal distribution of strength, respectively.

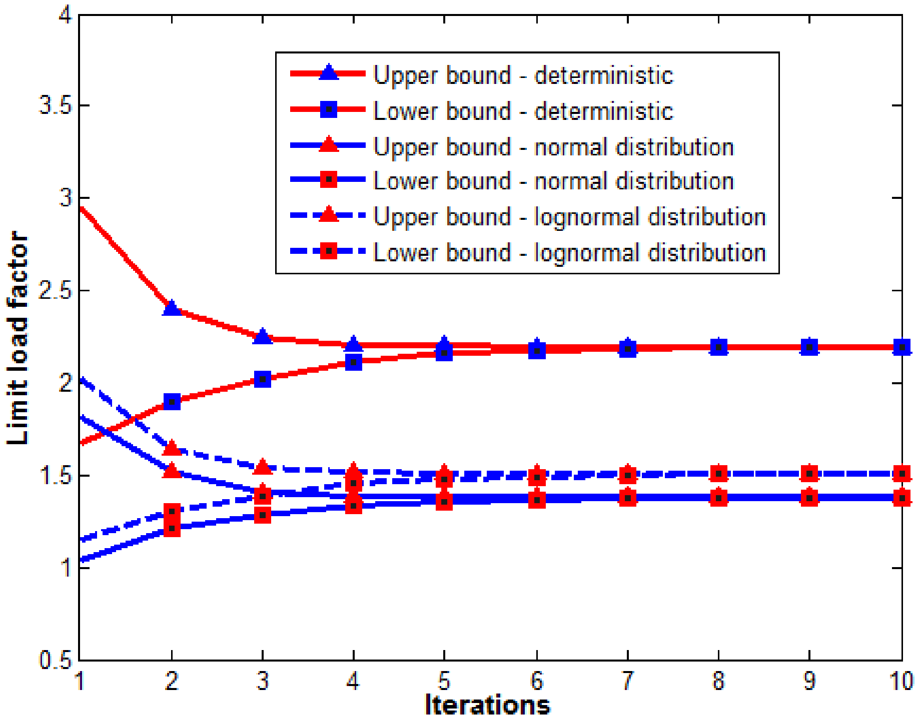


Figure 4. 2 Convergence of the limit load factor in case of deterministic and random strength.

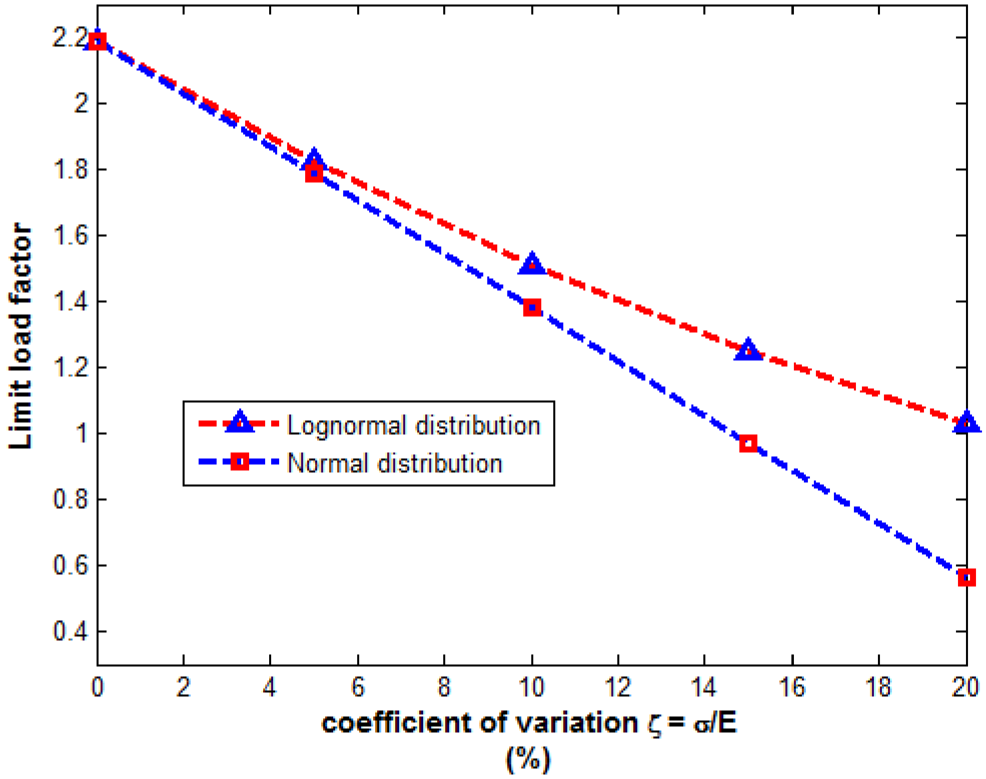


Figure 4. 3 Dependence of load the factor on the coefficient of variation ξ

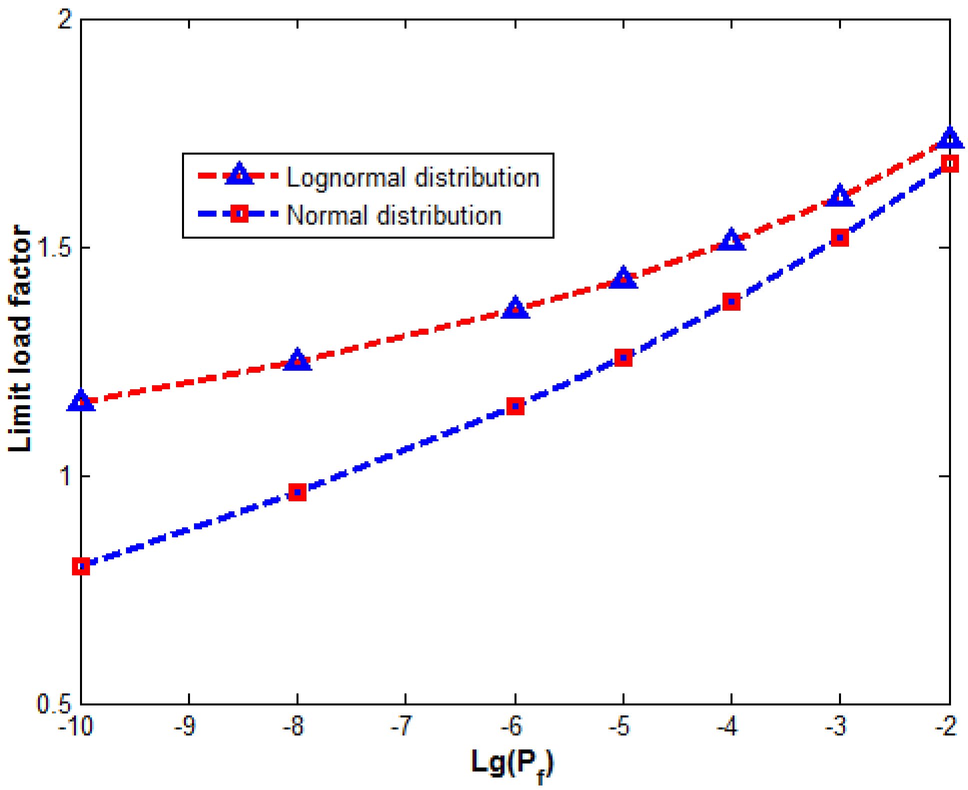


Figure 4. 4 Dependence of the load factor on the failure probability

The dependence of load factors on the coefficient of variation ξ and on failure probability are presented in figures 4.3, 4.4.

Analytical solution:

The limit load problem has an analytical solution. For the kinematic theorem, observe that the plastic moment of the first span is lower than that one of the second span and the applied load P_1 is greater than P_2. The failure mechanism is shown in Fig. 4.5.

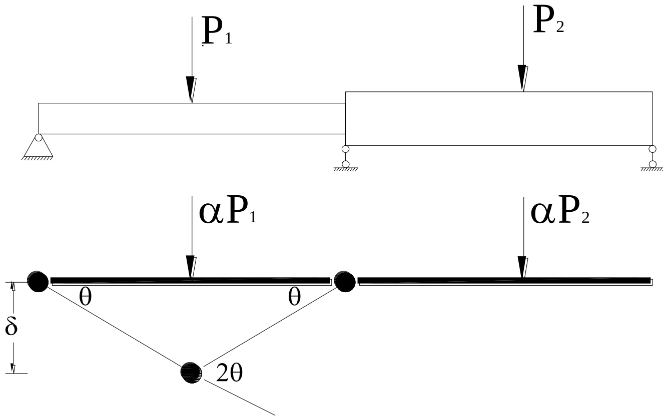


Figure 4. 5 The failure mechanism of the beam at limit state.

We easily calculate the upper bound limit load factor from the virtual work equation:

$$\alpha P_1 \cdot \delta = M_{0,1} \cdot 2\theta + M_{0,1} \cdot \theta \tag{4.1}$$

$$\begin{aligned} \alpha P_1 \cdot \delta &= 2\frac{\delta}{L} M_{0,1} + M_{0,1}\frac{\delta}{L} = 3M_{0,1}\frac{\delta}{L} \\ \Rightarrow \alpha &= \frac{3M_{0,1}}{P_1 L} = \frac{3 \cdot 2\,\mathrm{kNm}}{3\,\mathrm{kN} \cdot 1\,\mathrm{m}} = 2 \;. \end{aligned} \tag{4.2}$$

This is the exact limit load because the static theorem has the same result.

For random plastic moment, we can replace $M_{0,1}$ by $\overline{M}_{0,1}$ in the deterministic equivalent problem,

where $\overline{M}_{0,1} = \mu_1 - \kappa\sigma_1$ for normal distribution, $\overline{M}_{0,1} = e^{\mu_1 - \kappa\sigma_1}$ for lognormal distribution.

+ If the plastic moment $M_{0,1}$ is normally distributed with $\mathrm{E}\left[M_{0,1}\right] = 2\,\mathrm{kNm}$ and $\mathrm{Var}\left(M_{0,1}\right) = \left(0.1m\right)^2 = \left(0.2\,\mathrm{kNm}\right)^2$ then limit load factor can be computed from the virtual work equation (4.1):

$$\alpha_{\lim} = \frac{3\overline{M}_{0,1}}{P_1 L} = \frac{3\cdot(2-3.719\cdot 0.2)\,\mathrm{kNm}}{3\mathrm{kN}\cdot 1\mathrm{m}} = 1.256$$

+ If $M_{0,1}$ be lognormally distributed with $\mathrm{E}\left[M_{0,1}\right] = 2\,\mathrm{kNm}$ and $\mathrm{Var}\left(M_{0,1}\right) = (0.1m)^2 = (0.2\,\mathrm{kNm})^2$, respectively. Then the parameters of the lognormal distribution are computed as:

$$\mu = \ln\left[\frac{\mathrm{E}\left[M_{0,1}\right]}{\sqrt{\dfrac{\mathrm{Var}\left(M_{0,1}\right)}{\mathrm{E}^2\left[M_{0,1}\right]}+1}}\right] = \ln\left[\frac{2}{\sqrt{\dfrac{0.2^2}{2^2}+1}}\right] = 0.6882$$

$$\sigma = \sqrt{\ln\left(\frac{\mathrm{Var}\left(M_{0,1}\right)}{\mathrm{E}^2\left[M_{0,1}\right]}+1\right)} = \sqrt{\ln\left(\frac{0.04}{4}+1\right)} = 0.0998$$

For the chosen reliability level ($\kappa = 3.719$) the limit load factor is:

$$\alpha = \frac{3e^{\mu-\kappa\sigma}}{P_1 L} = \frac{3e^{0.6882-3.719\cdot 0.0998}}{3\cdot 1} = 1.373\,.$$

In table 4.1 our results in comparison with the results of Sikorski and Borkowski [64] The limit loads in [64] and the analytical limit loads are based on beam theory and are therefore different from the numerical limit loads which are based on plane stress FEM discretization.

Table 4. 1 Limit load factor of the two-span beam

Lower bound -	Upper bound -	Method, reference
2.19 (deterministic)	2.19 (deterministic)	numerically [149]
2 (deterministic)	2 (deterministic)	analytically [149]
1.15 (normal)	1.36 (normal)	[64]
1.38 (normal)	1.38 (normal)	numerically [149]
1.256 (normal)	1.256 (normal)	analytically [149]
1.509 (lognormal)	1.509 (lognormal)	numerically
1.373 (lognormal)	1.373 (lognormal)	analytically

- *Random loads, strength is deterministic:*

We study the situation at which strength is deterministic, loads acting on the beam are random with the coefficient of variation $\xi = 0.05$. The algorithm A3 is used to find the limit load. Figure 4.6 shows the convergence of limit load factors using algorithms A1, A2, A3. Dual solution and upper bound solution converge to 2.0 for deterministic analysis. For normally distributed loads, the upper bound solution shows a limit load factor 1.855 (with the reliability level 0.9999). The dependence of the limit load factor on the coefficient of variance or on reliability are also investigated (figures 4.7, 4.8)

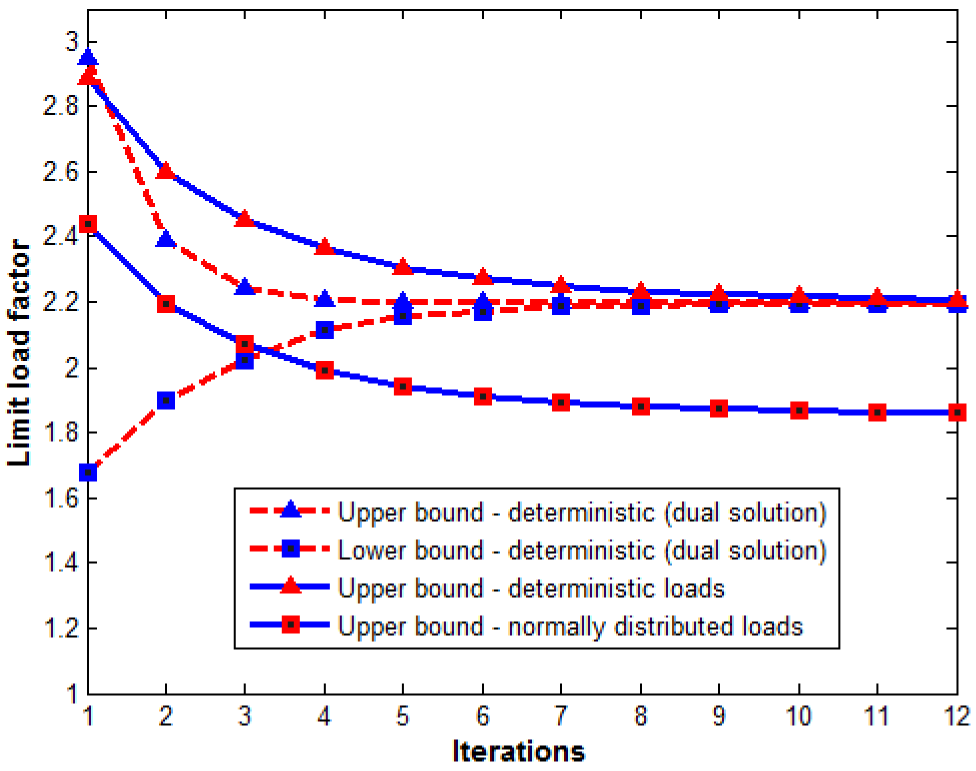


Figure 4. 6 Convergence of limit load factors in case of random load

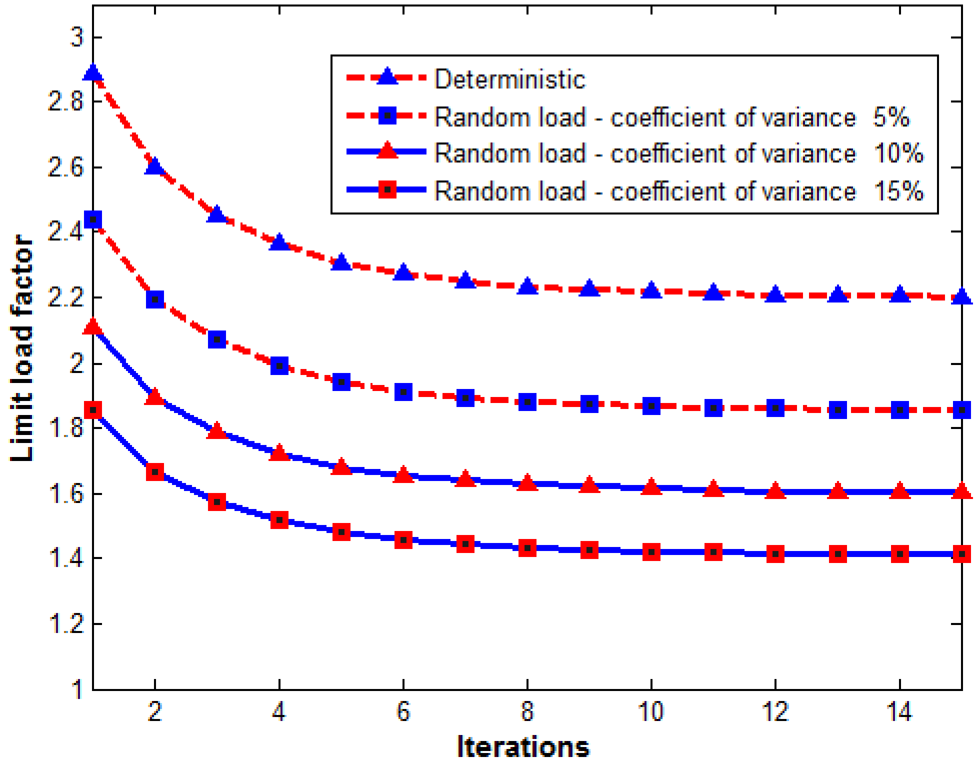


Figure 4. 7 Limit loads with different coefficient of variance of random load

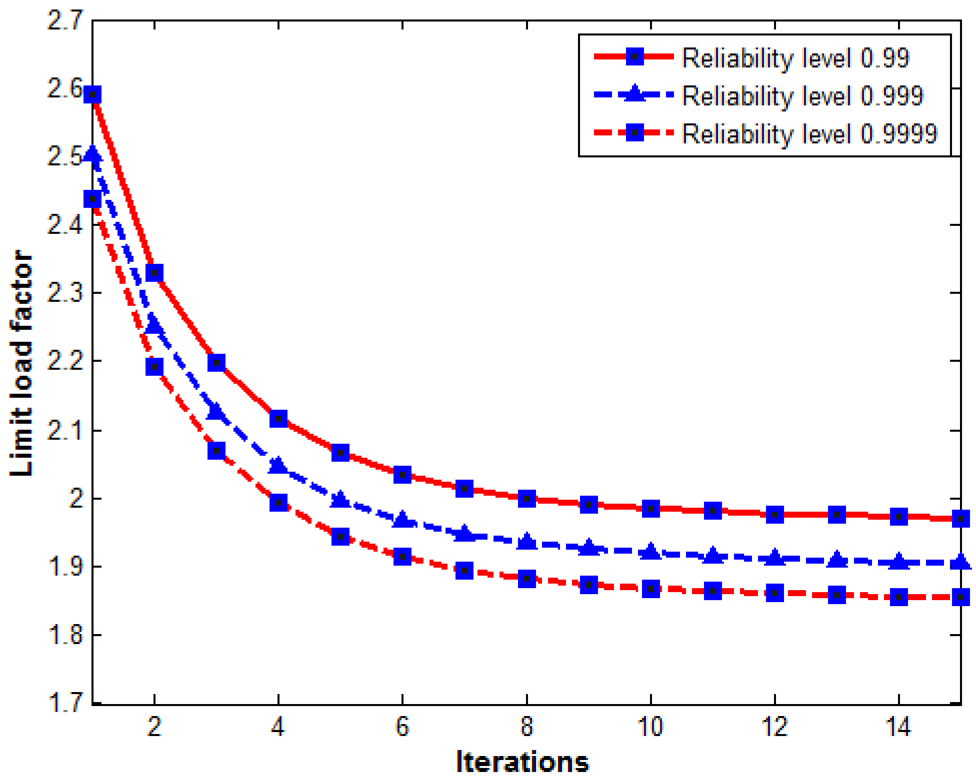


Figure 4. 8 Limit loads with difference of reliability level of random load (strength is deterministic)

Similar to the situation of random strength, the analytical solution can be obtained. We can replace P_1 by $\mu_{P_1} + \kappa\sigma_{P_0} = \mu_{P_1}\left(1+\kappa\nu_{P_1}\right)$ in the deterministic equivalent problem by using (4.1). Let P_1 be normally distributed with mean value $\mu_{P_1} = 3\,\mathrm{kN}$ and standard deviation $\sigma_{P_1} = 0.3\,\mathrm{kNm}$, respectively, and strength deterministic. For the chosen reliability level the limit load factor is:

$$\alpha^{+}_{\lim} = \frac{3M_{0,1}}{\mu_{P_1}\left(1+\kappa\nu_{P_1}\right)\cdot L} = \frac{3\cdot 2\ \mathrm{kNm}}{3\ \mathrm{kN}\left(1+3.719\cdot 0.1\right)\cdot 1\ \mathrm{m}} = 1.4578$$

$$\alpha^{+}_{\lim} = \frac{3M_{0,1}}{\mu_{P_1}\left(1+\kappa\nu_{P_1}\right)\cdot L} = \frac{3\cdot 2\ \mathrm{kNm}}{3\ \mathrm{kN}\left(1+3.719\cdot 0.05\right)\cdot 1\ \mathrm{m}} = 1.6864$$

- *Random strength and loads*:

We now consider the problem with random load and random strength. Load is distributed normally and strength is random with normal /lognormal distributions. The figure 4.9 shows upper bound solutions for 3 case: deterministic, normally distributed load and normally distributed strength, normally distributed load and lognormally distributed strength. In figure 4.10 we see a picture showing the evolution of limit load factor in four situations (with the coefficient of variation for load 5% and 10% for strength):

+ Deterministic of strength and load: 2.19

+ Strength is deterministic, load random: 1.89

+ Load is deterministic, strength is random (normal distribution): 1.38

+ Both of load and strength are random: 1.19

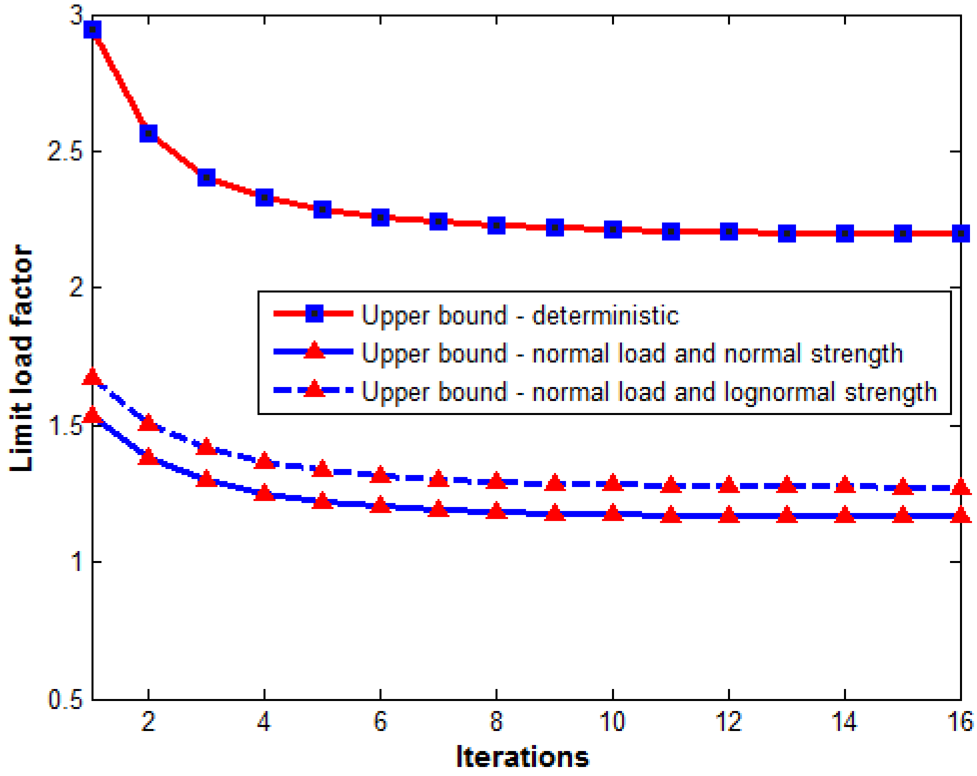


Figure 4. 9 Limit load factors with normal load and some distributions of strength

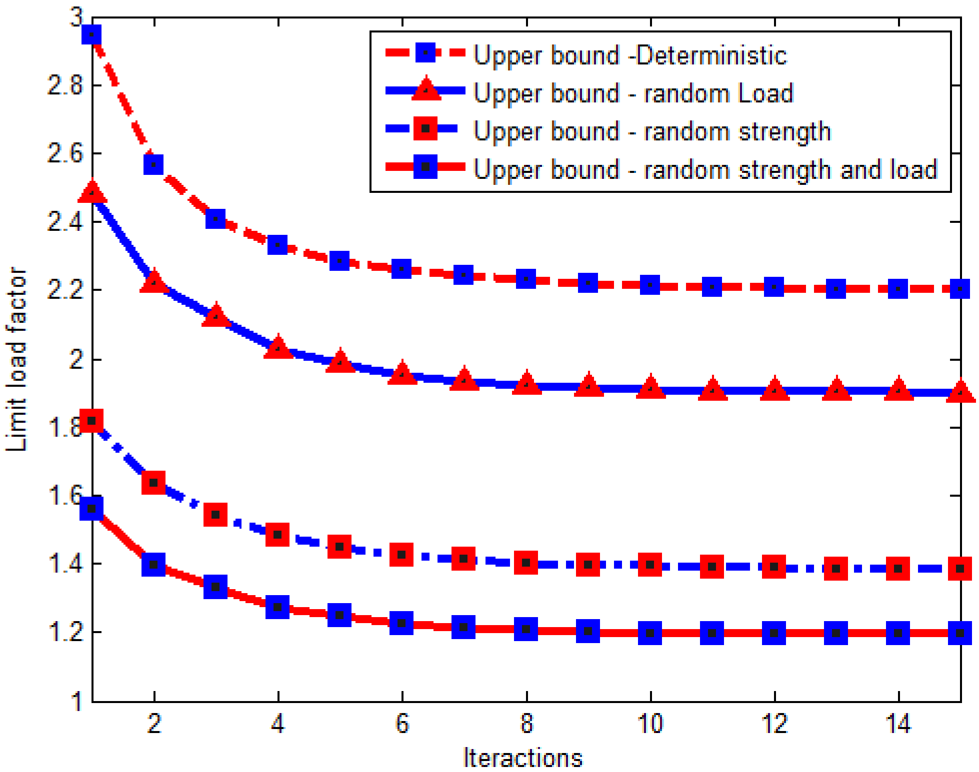


Figure 4. 10 Limit load factors with random load and strength (normal distribution)

The analytical solution for limit load factor can be calculated. From (4.1) we replace

$M_{0,1}$ by $\overline{M}_{0,1} = \mu_{M_{01}} - \kappa\sigma_{M_{01}} = \mu_{M_{01}}(1-\kappa\nu_{M_{01}})$ and P_1 by $\overline{P}_1 = \mu_{P_1} + \kappa\sigma_{P_1} = \mu_{P_1}\left(1+\kappa\nu_{P_1}\right)$

$$\alpha_{\lim}^{+} = \frac{3\mu_{M_{01}}(1-\kappa\nu_{M_{01}})}{\mu_{P_1}\left(1+\kappa\nu_{P_1}\right)\cdot L} = \frac{3\cdot 2\left(1-3.719\cdot 0.1\right)\ \text{kNm}}{3\ \text{kN}\left(1+3.719\cdot 0.05\right)\cdot 1\ \text{m}} = 1.059$$

Table 4. 2 Comparison with results of Sikorski and Borkowski [64]

	Sikorski and Borkowski [64]		Present (upper bound)	
situations	Lower bound	Upper bound	Analytical solution	Numerical solution
Deterministic	2	2	2	2.19
Random R	1.15	1.36	1.256	1.38
Random L	1.22	1.64	1.686	1.89
Random R+L	1.08	1.33	1.059	1.19

Figure 4.11 shows the relation between limit load factor and failure probability in different situations,

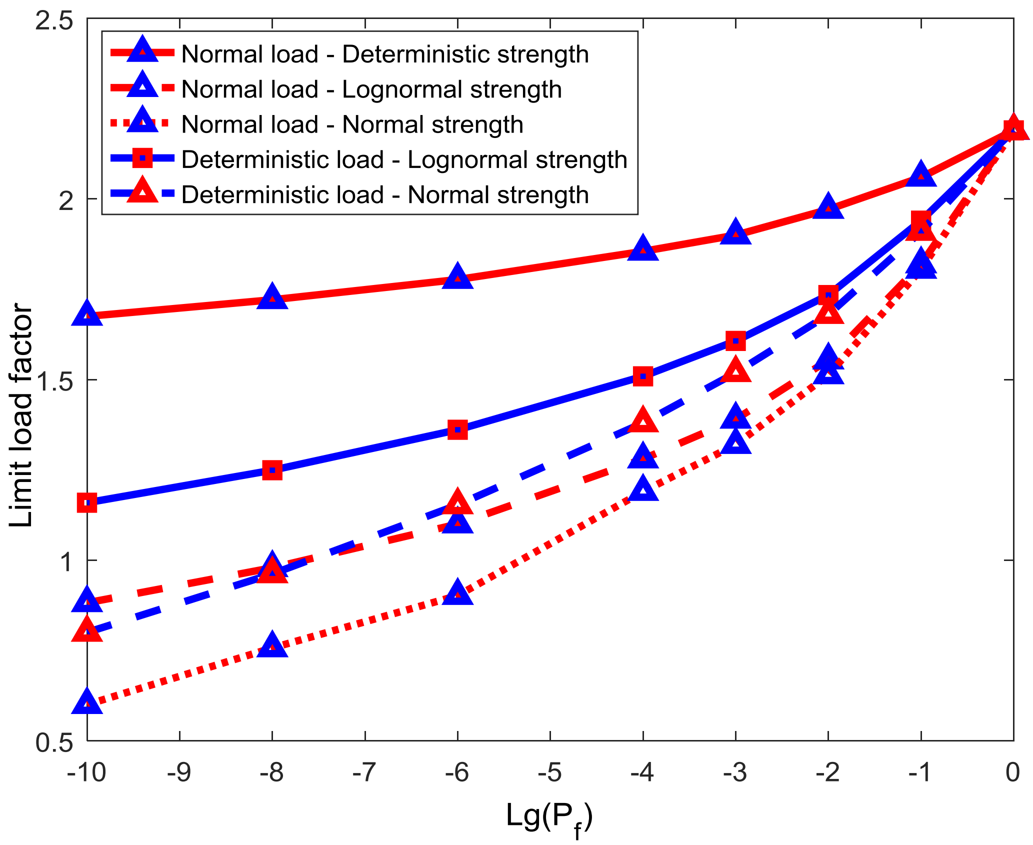


Figure 4. 11 Dependence of the load factor on the failure probability in different situations

Note: To advoice the stress singularities at the point loads, we can replace the point forces by stacially equivalent distributed forces as shown in Figure 4.12. Table 4.3 shows the comparison between cases for point forces and distributed forces for some situations of random strength. It is shown that they are similar

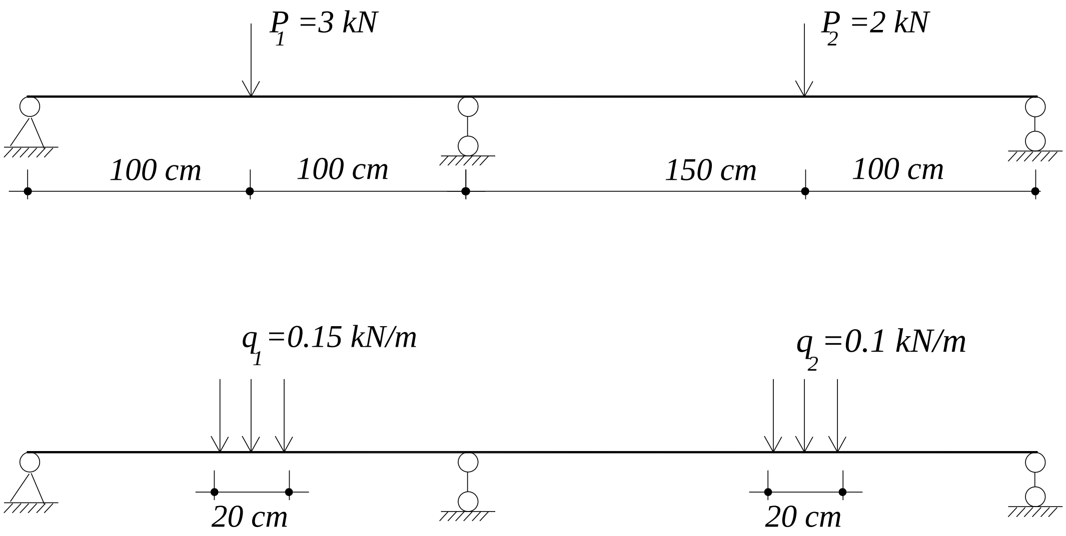


Figure 4. 12 Point forces are replaced by distributed forces

Table 4. 3 Limit load factors in comprison for two case: point forces and distributed forces

Situations	Point forces	Distributed forces
Deterministic	2.19	2.24
Normal distribution	1.38	1.40
Lognormal distribution	1.51	1.54

4.2. Simple frame

In the second example, we investigate a simple frame which is depicted in Figure 4.11 (a,b). The frame carries uniormly distributed loads which can vary independently in the load domain as shown in figure 4.12. The loads are considered as random variables which are considered to be distributed normally. The geometrical and material data are chosen as in [133], i.e. $E = 2 \cdot 10^5$ MPa , $\nu = 0.3$, and $\sigma_y = 10$ MPa . $p_1 \in [1.2, 3.0]$ and $p_2 \in [0.4, 1.0]$. We investigate limit and shakedown loads for two cases:

[1] The left side of beam component can move only in horizontal direction
[2] The left side of beam component is clamped

The frame is discritized by 1600 smoothed T3 elements as shown in Fig. 4.12.

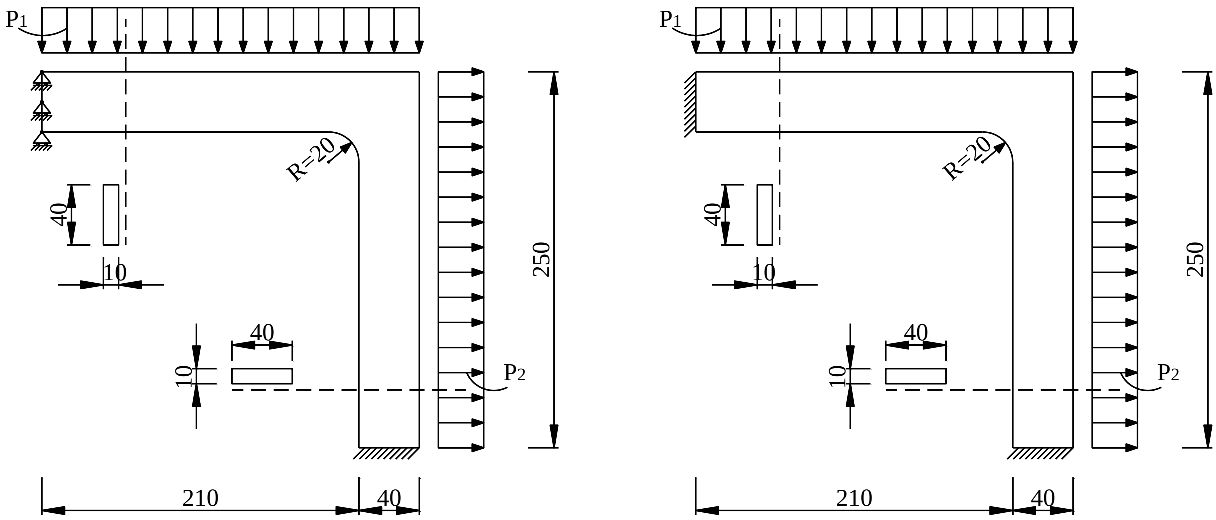


Figure 4. 13 The geometrical dimensions

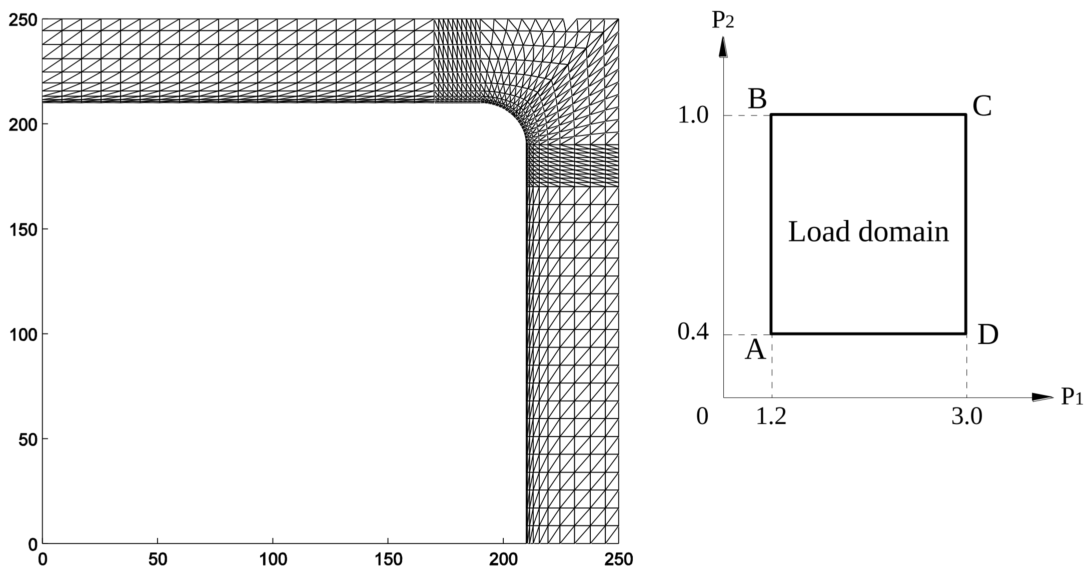


Figure 4. 14 The geometrical dimensions and FE-mesh and load domain.

- **The strength of frame is random, the load acting on frame is deterministic**.

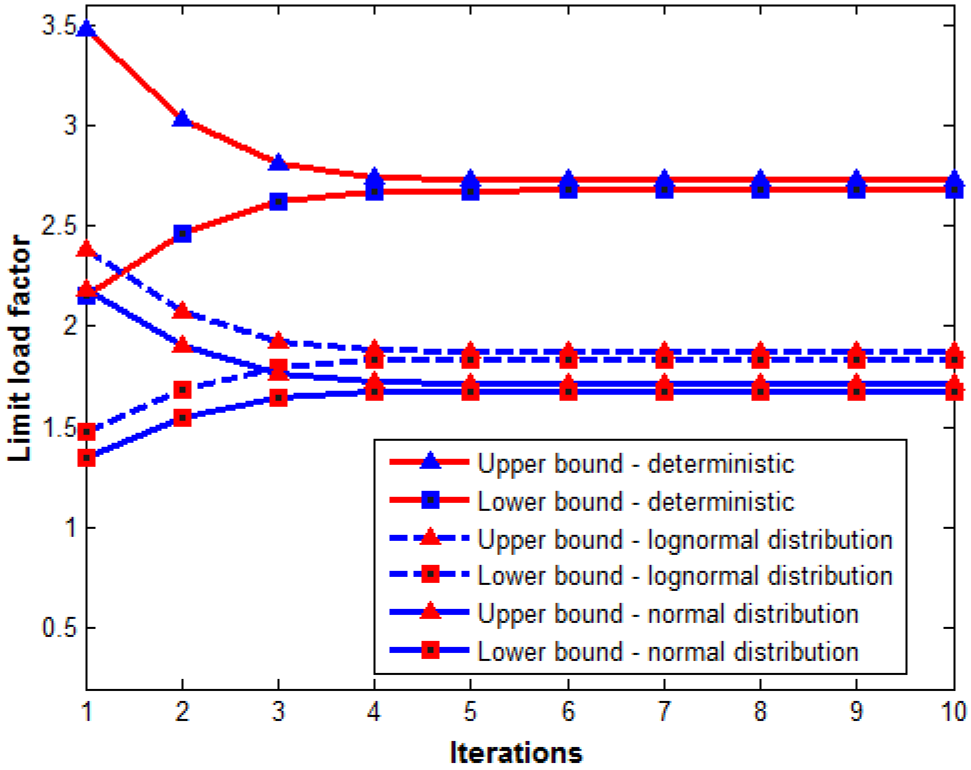


Figure 4. 15 Limit load factor with random strength, deterministic loads.

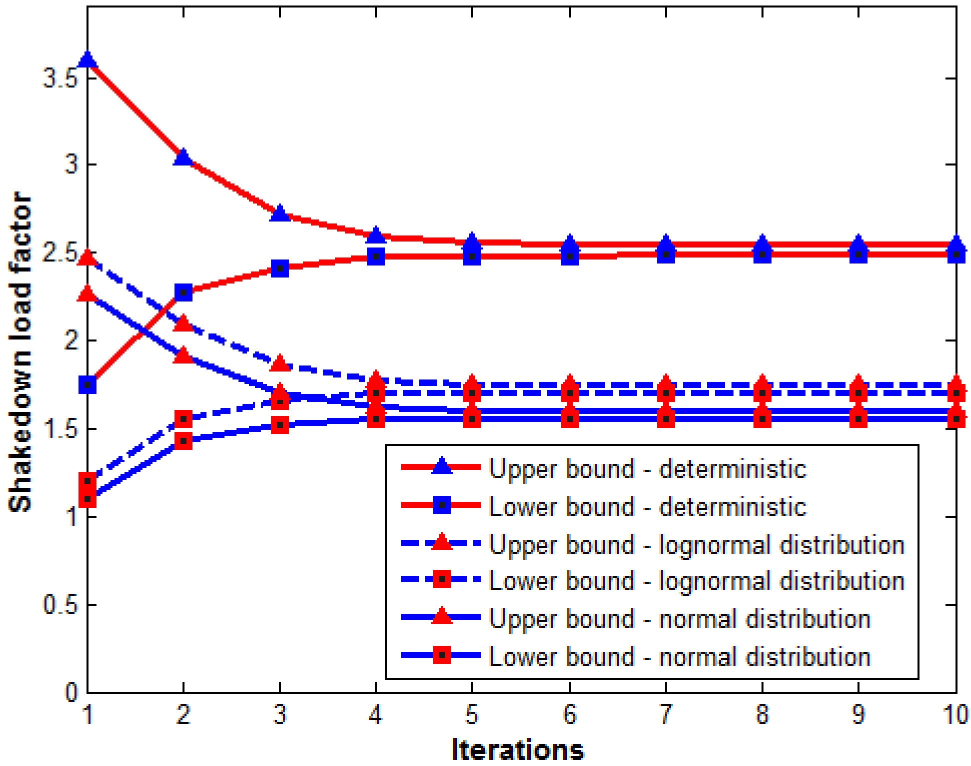


Figure 4. 16 Shakedown load factor with random strength, deterministic loads.

Table 4. 4 Limit analysis: comparison case a (Fig 4.11a)

	Garcea *et al*. [133]	Present		
(p_1, p_2)	Deterministic	Deterministic	Normal	Lognormal
(1.2, 1.0)	2.975	2.930	1.793	1.963
(3.0, 0.4)	2.831	2.985	1.856	2.045
(3.0, 1.0)	2.645	2.705	1.697 [31]	1.856

Table 4. 5 Limit analysis: comparison case b (Fig 4.11b)

	Garcea *et al*. [133]	Present		
(p_1, p_2)	Deterministic	Deterministic	Normal	Lognormal
(1.2, 1.0)	7.804	8.099	5.056	5.583
(3.0, 0.4)	4.207	4.365	2.730	3.002
(3.0, 1.0)	3.949	4.072	2.541	2.793

Table 4. 6 Shakedown analysis: comparison case a (Fig 4.11a), $(p_1, p_2) = (3.0,\ 1.0)$

	Garcea *et al.* [133]	Present		
Limits	Deterministic	Deterministic	Normal	Lognormal
Elastic	1.203	1.192	0.749	0.819
Alternating	2.940	2.922	1.835	2.006
Ratcheting	2.473	2.521	1.582 [31]	1.730

Table 4. 7 Shakedown analysis: comparison case b (Fig 4.11b), $(p_1, p_2) = (3.0,\ 1.0)$

	Garcea *et al*. [133]	Present		
Limits	Deterministic	Deterministic	Normal	Lognormal
Elastic	1.355	1.427	0.896	0.979
Alternating	4.518	4.757	2.987	3.266
Ratcheting	3.925	4.051	2.558	2.784

Figure 4.15 and 4.16 show the evolutions of limit and shakedown load factors for case (a) for both situations: deterministic and random strength. For limit analysis with $p_1 = 3.0,\ p_2 = 1.0$, all the two bounds converge to the solutions $\alpha_{\lim} = 2.705$ in case of deterministic strength and 1.856 in case of lognormally distributed strength. For the shakedown analysis, the results give the shakedown load factors $\alpha = 2.521$ and $\alpha = 1.730$ corresponding to deterministic and

lognormally distributed random strength, respectively. Tables 4.4 - 4.7 present results in comparison with deterministic results of Garcea *et al.* [133].

- **The load on frame is random, strength of the frame is deterministic**

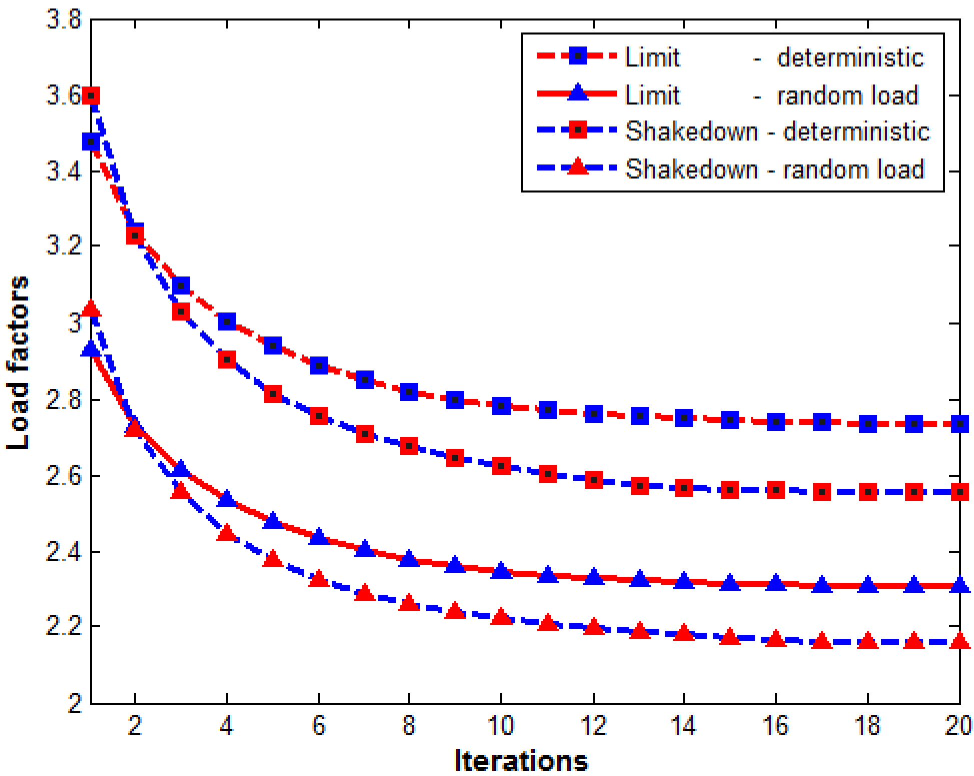


Figure 4. 17 Covergence of limit and shakedown load factors for case a

Table 4. 8 Limit analysis: comparison case a (Fig 4.11a)

	Garcea *et al.* [133]	Tran *et al.* [26]	Present	
(p_1, p_2)	Deterministic	Deterministic	Deterministic	Radom load
(1.2, 1.0)	2.975	2.970	2.937	2.476
(3.0, 0.4)	2.831	2.792	3.045	2.569
(3.0, 1.0)	2.645	2.659	2.731	2.304

Table 4. 9 Limit analysis: comparison case b (Fig 4.11b)

	Garcea *et al.* [133]	Tran *et al.* [26]	Present	
(p_1, p_2)	Deterministic	Deterministic	Deterministic	Radom load
(1.2, 1.0)	7.804	7.901	8.099	6.867
(3.0, 0.4)	4.207	4.241	4.365	5.056
(3.0, 1.0)	3.949	4.008	4.072	4.311

Table 4. 10 Shakedown analysis: comparison case a (Fig 4.11a),

Limits	Garcea *et al.* [133]	Tran *et al.* [26]	Present	
	Deterministic	Deterministic	Deterministic	Radom load
Elastic	1.203	1.192	1.192	1.192
Alternating	2.940	2.922	2.922	2.922
Ratcheting	2.473	2.487	2.521	2.153

Table 4. 11 Shakedown analysis: comparison case b (Fig 4.11b),

Limits	Garcea *et al.* [133]	Tran *et al.* [26]	Present	
	Deterministic	Deterministic	Deterministic	Radom load
Elastic	1.355	1.427	1.427	1.427
Alternating	4.518	4.657	4.757	4.757
Ratcheting	3.925	4.006	4.051	3.455

Figure 4.17 shows the evolutions of limit and shakedown load factors for case (a) in the situations: deterministic and random loads. For limit analysis with $p_1 = 3.0,\ p_2 = 1.0$, the upper bound converge to the solutions $\alpha^+_{\lim} = 2.731$ in case of deterministic and 2.304 in the case that the load is distributed normally.

For shakedown analysis, the load domain is defined by

$$p_1 \in [1.2 \quad 3.0],\ p_2 p_1 \in [0.4 \quad 1.0].$$

The results give the shakedown load factors $\alpha^+_{sd} = 2.521$ and $\alpha^+_{sd} = 2.153$ corresponding to deterministic and random strength, respectively. Tables 3-6 present results in comparison with deterministic results of Garcea *et al.* [133] and Tran *et al.* [26]

- **Random strength and random loads**
  Now we investigate limit and shakedown load factors in situation at which strength and load are distributed normally. The evolution of limit and shakedown load factors of case a and case b are shown in figures (4.18-4.21). Tables (4.12-4.15) list resutls in situations compared with some authors.

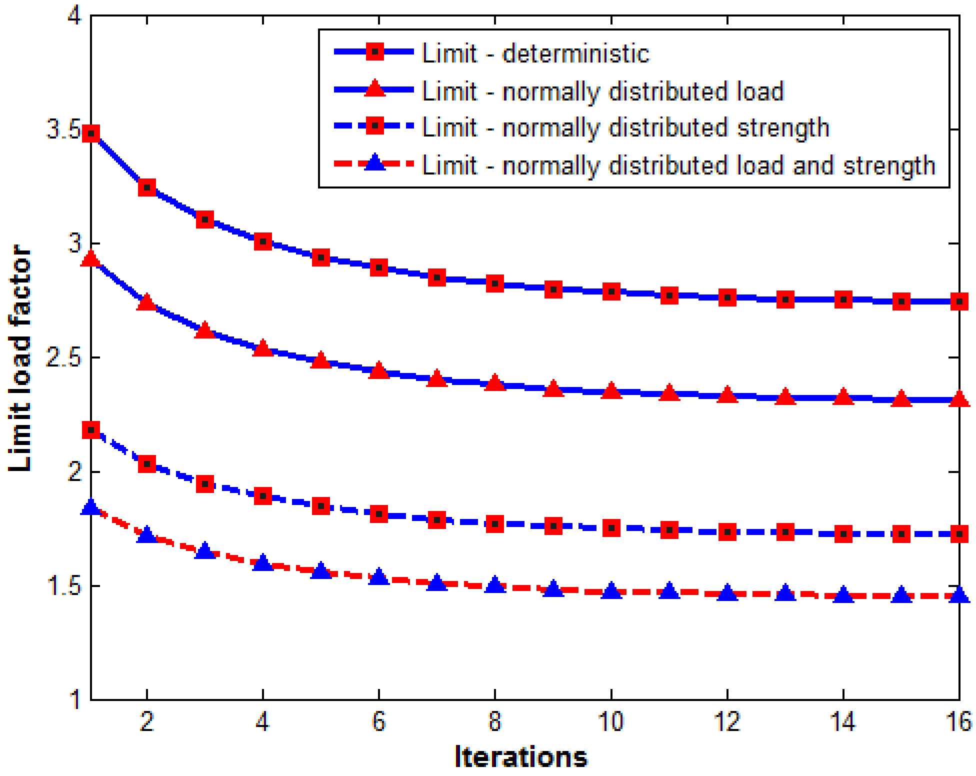


Figure 4. 18 Convergence of limit load factor for case_a

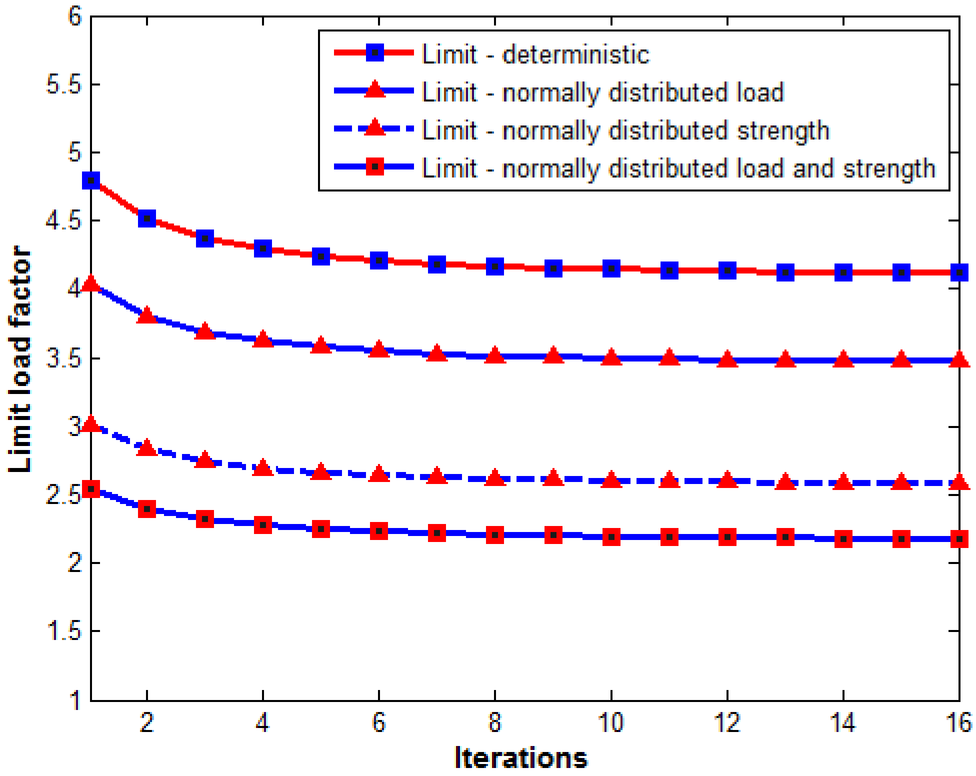


Figure 4. 19 Convergence of limit load factor for case_b

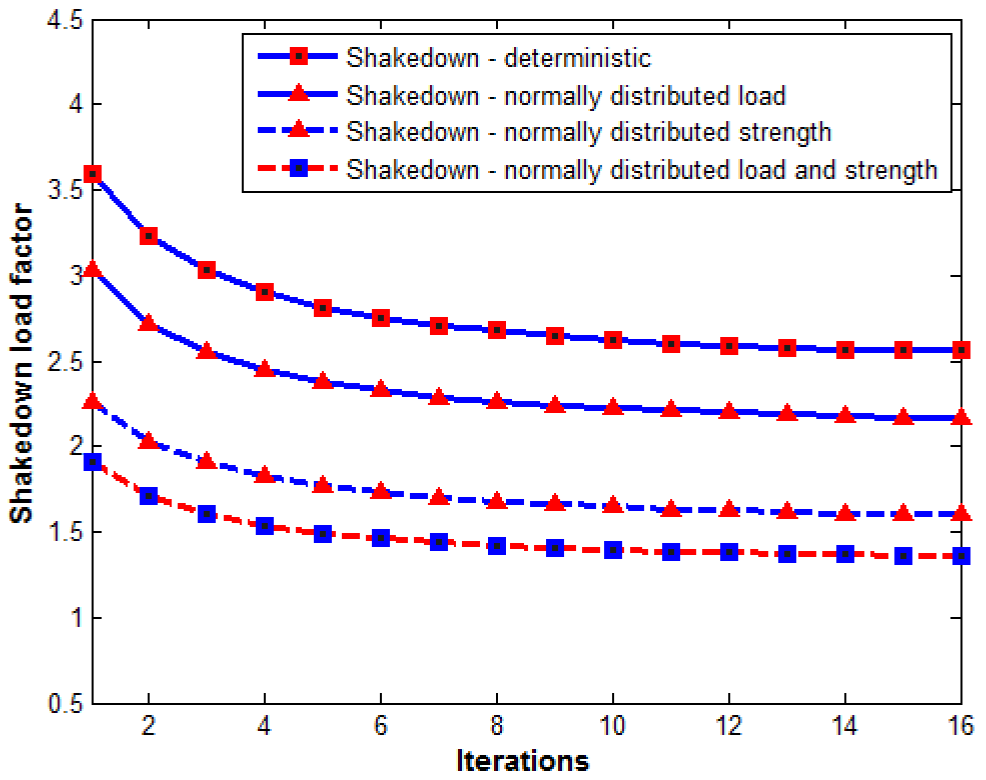


Figure 4. 20 Convergence of shakedown load factor for case_a

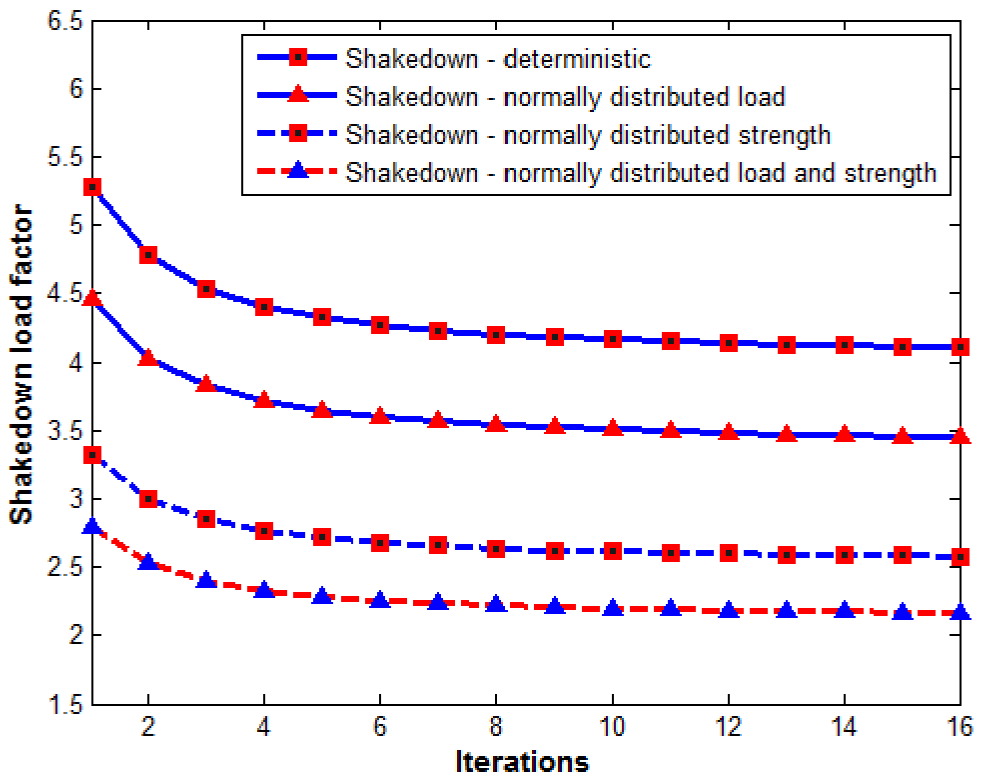


Figure 4. 21 Convergence of shakedown load factor for case_b

Table 4. 12 Limit analysis: comparison for case a (Fig. 4.11a)

	(1.2, 1.0)	(3.0, 0.4)	(3.0, 1.0)	
Garcea *et al*. [133]	2.975	2.831	2.645	Deterministic
Tran *et al*. [26]	2.970	2.792	2.659	Deterministic
Present	2.930	2.985	2.705	Deterministic
	2.476	2.569	2.304	Random load
	1.793	1.856	1.697	Random strength
	1.554	1.610	1.447	Random load and strength

Table 4. 13 Limit analysis: comparison for case b (Fig.4.11b)

	(1.2, 1.0)	(3.0, 0.4)	(3.0, 1.0)	
Garcea *et al*. [133]	7.804	4.207	3.949	Deterministic
Tran *et al*. [26]	7.901	4.241	4.008	Deterministic
Present	8.099	4.365	4.072	Deterministic
	6.867	3.696	3.468	Random load
	5.056	2.730	2.541	Random strength
	4.311	2.323	2.176	Random load and strength

Table 4. 14 Shakedown analysis: comparison for case a (Fig.4.11a)

	Elastic limit	Alternating	Ratcheting	Situations
Garcea *et al*. [133]				Deterministic
Tran *et al*. [26]				Deterministic
Present	1.192	2.922	2.521	Deterministic
	1.192	2.922	2.153	Random load
	0.749	1.835	1.582	Random strength
	0.749	1.835	1.360	Load and strength

Table 4. 15 Shakedown analysis: comparison for case b (Fig.4.11b)

	Elastic limit	Alternating	Ratcheting	Situations
Garcea *et al*. [133]	1.355	4.518	3.925	Deterministic
Tran *et al*. [26]				Deterministic
Present	1.427	4.757	4.051	Deterministic
	1.427	4.757	3.455	Random load
	0.896	2.987	2.558	Random strength
	1.835	2.988	2.162	Load and strength

4.3. Square plate with a central circular hole

In this example, we consider a classic problem in numerical shakedown analysis as shown in Figure 4.22. The ratio between the diameter D of the hole and the length L of the plate is $D/L = 0.2$. The plate is subjected to two uniform loads p_1, p_2 varying independently. The loads are considered as random varialbes distributed normally. The reliability level is $\psi = 0.9999$ so that $\kappa = 3.719$. Let investigate limit load factor and shakedown load factor under two situations: uniaxial loading and equibiaxial loading.

In this analysis, due to the symmetry, one fourth of the plate is modelled and discretized by 288 smoothed T3 elements as figure 4.22b

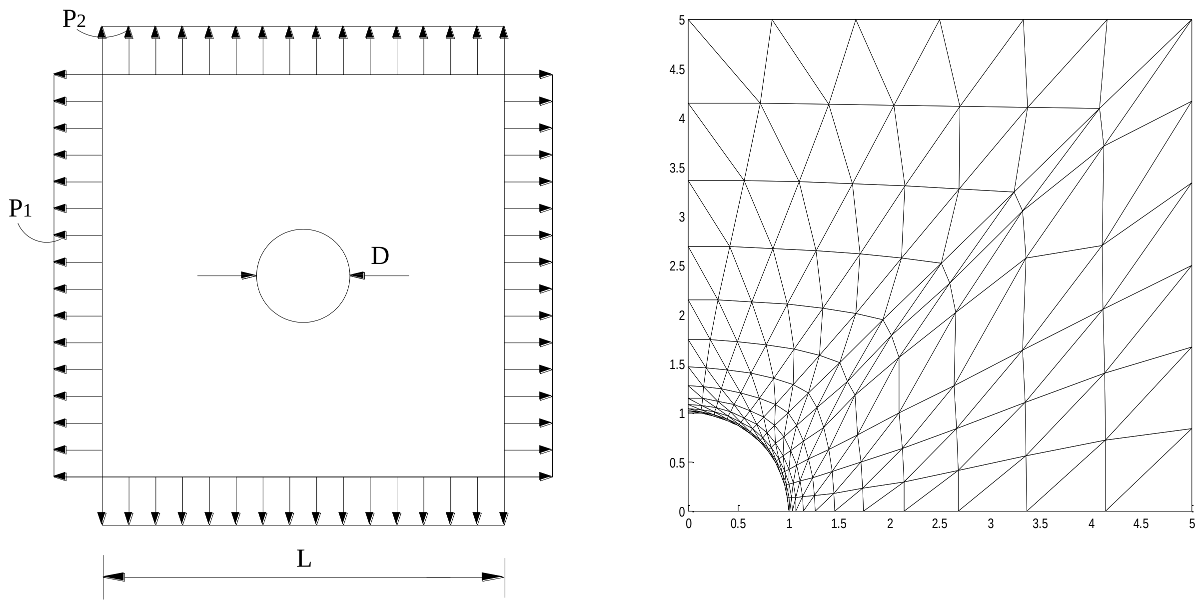


Figure 4. 22 Square plate with hole ($D/L = 0.2$) subjected uniform loads.

- **Random strength, deterministic loads**

The exact limit load under uniaxial loading ($p_2 = 0$) is $p_1 = p_L = (1 - D/L)\sigma_y$, [32], so that the analytical limit load factor is

$$\alpha_{\lim} = \frac{p_L}{\sigma_y} = 1 - \frac{D}{L} = 0.8 \tag{4.3}$$

Let the mean value $\mathrm{E}\left[\sigma_y\right] = 1\,\mathrm{kNm}^{-1}$ and standard deviation $\mathrm{Var}\left(\sigma_y\right) = (0.1\mu)^2 = \left(0.1\,\mathrm{kNm}^{-1}\right)^2$. For $\sigma_y \sim N(\mu,\sigma^2)$ with $\mu = \mathrm{E}\left[\sigma_y\right]$ and $\sigma^2 = \mathrm{Var}\left(\sigma_y\right)$

$$\alpha_{\lim} = \frac{p_L}{\sigma_y} = 0.8(\mu - \kappa\sigma) = 0.8(1 - 3.719 \cdot 0.1) = 0.502 .$$

For $\ln \sigma_y \sim N(\mu, \sigma^2)$, the analytical solution can be computed

$$\alpha_{\lim} = \frac{p_L}{\sigma_y} = 0.8 e^{(\mu - \kappa\sigma)} \tag{4.4}$$

in which the parameters of lognormal distribution are:

$$\begin{aligned}
\mu &= \ln\left[\frac{\mathrm{E}\left[\sigma_y\right]}{\sqrt{\dfrac{\mathrm{Var}\left(\sigma_y\right)}{\mathrm{E}^2\left[\sigma_y\right]}+1}}\right] = \ln\left[\frac{1}{\sqrt{\dfrac{0.1^2}{1^2}+1}}\right] = -4.975\cdot 10^{-3} \\
\sigma &= \sqrt{\ln\left(\frac{\mathrm{Var}\left(\sigma_y\right)}{\mathrm{E}^2\left[\sigma_y\right]}+1\right)} = \sqrt{\ln\left(\frac{0.01}{1^2}+1\right)} = 9.975\cdot 10^{-2}
\end{aligned} \tag{4.5}$$

$$\alpha = \frac{p_L}{\sigma_y} = 0.8 e^{(-4.975\cdot 10^{-3} - 3.719\cdot 9.975\cdot 10^{-2})} = 0.5493 \tag{4.6}$$

Tables 4.16, 4.17 show our results and results found in literature obtained by different FEM discretization and approaches for equibiaxial tension ($p_1 \in [0,1],\ p_2 \in [0,1]$) and uniaxial tension ($p_1 \in [0,1],\ p_2 = 0$). Figures 4.23, 4.24 present the evolutions of limit and shakedown load factors under uniaxial tension. Figure 4.25 shows the interaction diagram of limit and shakedown analyses.

Table 4. 16 Limit analysis: comparison with some authors.

<table>
<tr><th>Tension loading</th><th>equibiaxial</th><th>uniaxial</th><th></th></tr>
<tr><td>Belytschko [151]</td><td>—</td><td>0.780</td><td rowspan="7">deterministic</td></tr>
<tr><td>Corradi et al. [152]</td><td>0.767</td><td>0.691</td></tr>
<tr><td>Genna [153]</td><td>—</td><td>0.793</td></tr>
<tr><td>Groß-Weege [154]</td><td>0.882</td><td>0.782</td></tr>
<tr><td>Garcea et al. [133]</td><td>0.902</td><td>0.806</td></tr>
<tr><td>Tran et al. [26]</td><td>0.896</td><td>0.797</td></tr>
<tr><td rowspan="3">Present</td><td>0.899</td><td>0.807</td></tr>
<tr><td>0.565</td><td>0.506</td><td>normal</td></tr>
<tr><td>0.618</td><td>0.554</td><td>lognormal</td></tr>
<tr><td>Gaydon, McCrum [155]</td><td>—</td><td>0.800</td><td>deterministic</td></tr>
<tr><td rowspan="2">Present:
analystical, exact</td><td>—</td><td>0.502</td><td>normal</td></tr>
<tr><td>—</td><td>0.549</td><td>lognormal</td></tr>
</table>

Table 4. 17 Shakedown analysis: comparison with some authors

Tension loading	equibiaxial	uniaxial	
Belytschko [151]	0.431	0.571	
Corradi *et al.* [152]	0.504	0.654	
Genna [153]	0.478	0.653	
Groß-Weege [154]	0.446	0.614	deterministic
Garcea *et al.* [133]	0.438	0.604	
Tran *et al.* [26]	0.434	0.601	
	0.436	0.602	
Present	0.274	0.378	normal
	0.299	0.414	lognormal

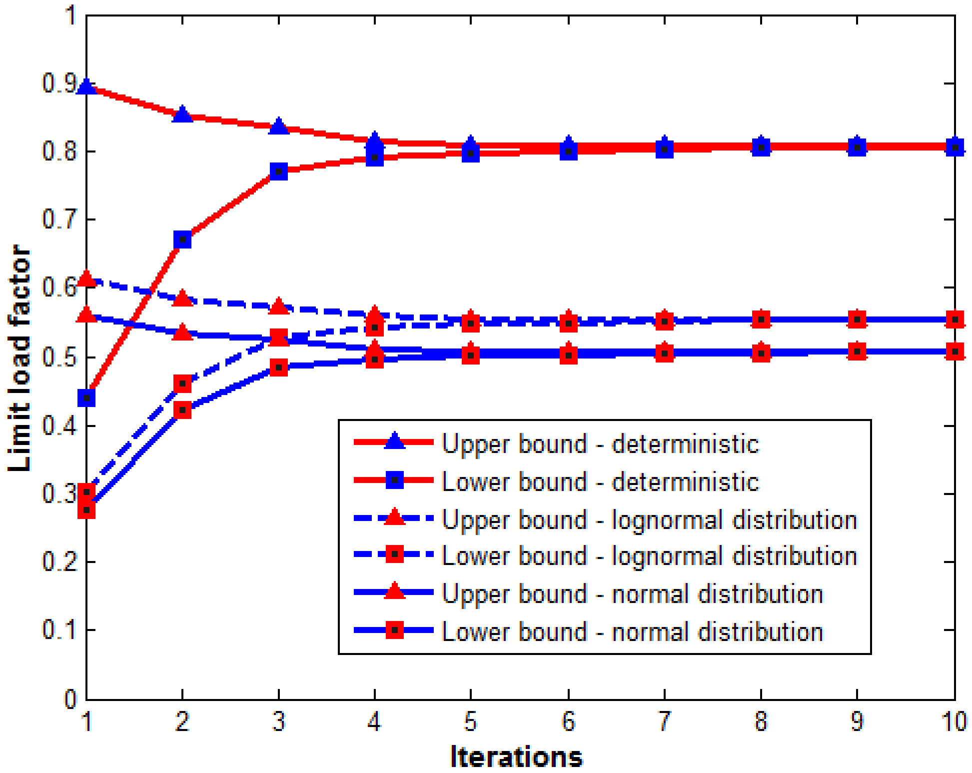


Figure 4. 23 Convergence of limit load factor in case of unaxial tension ($p_1 = 1,\ p_2 = 0$)

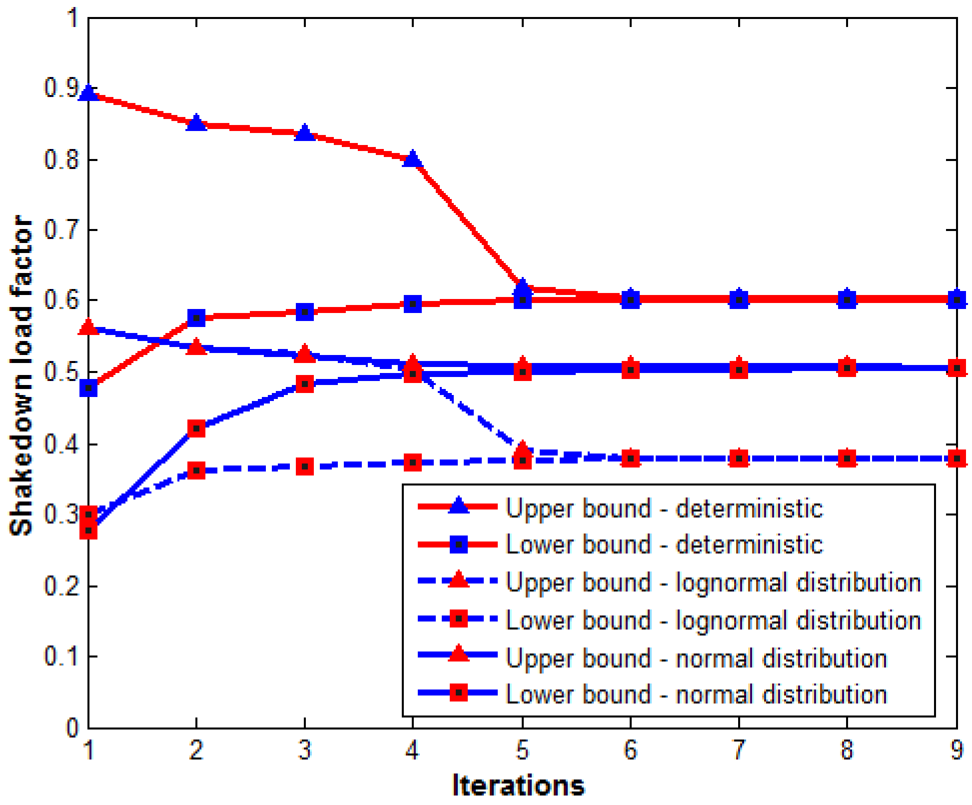


Figure 4. 24 Convergence of shakedown load factor ($p_1 \in [0,1]$, $p_2 \in [0,1]$)

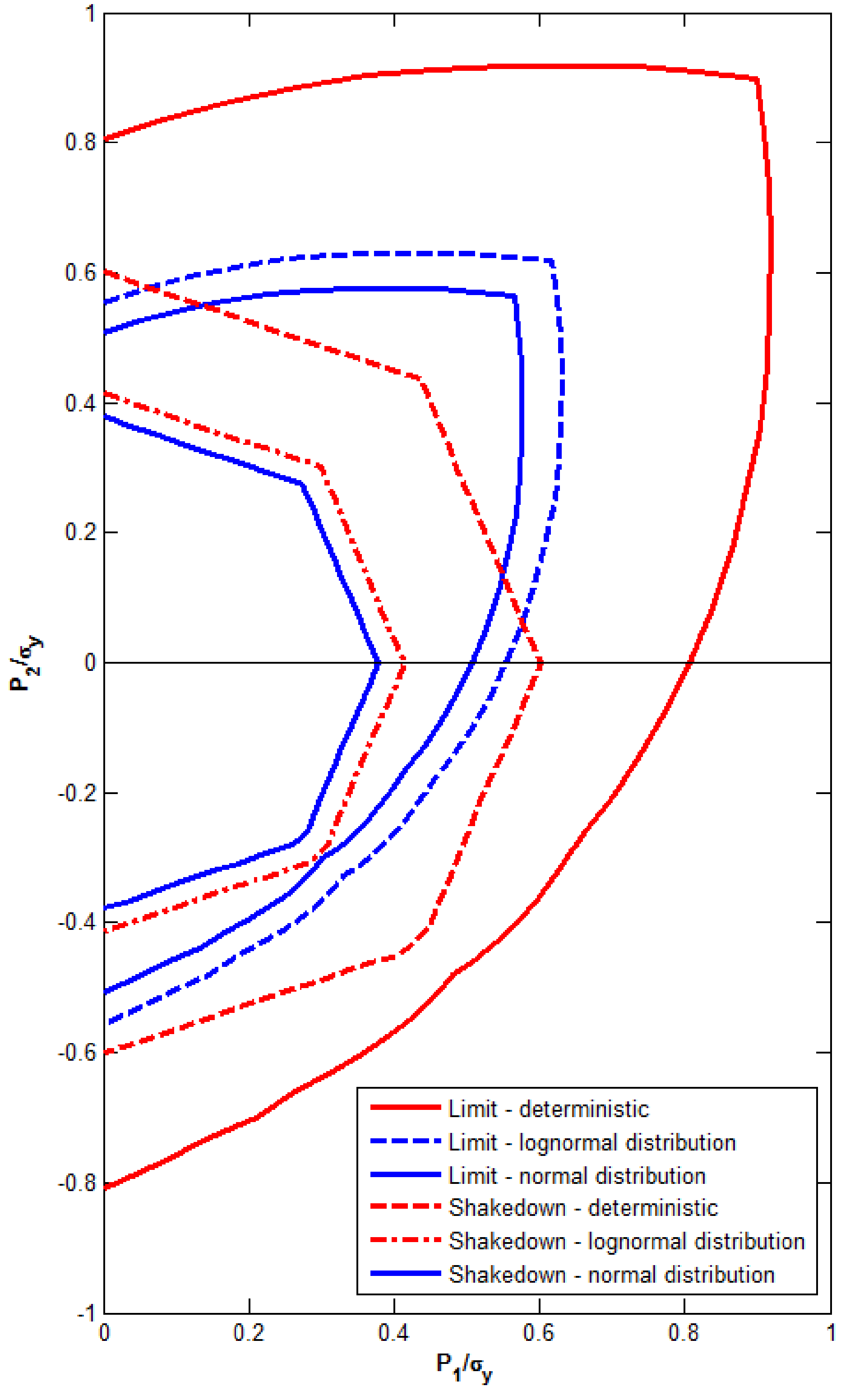


Figure 4. 25 Interaction diagram of square plate with central hole ($D/L = 0.2$). The diagramm can be extended symmetrically over all four quadrants.

- **Random load, deterministic strength**

The analytic limit load for the case of random load (coefficients of variation are 0.1 and 0.05 respectively) can be derived from (4.3):

$$\alpha_{\lim}^{+} = 0.8\frac{\sigma_y}{p} = 0.8\frac{\sigma_y}{\mu_p\left(1+\kappa\nu_p\right)} = 0.8\frac{\sigma_y}{\mu_p\left(1+3.719\cdot 0.1\right)} = 0.583\frac{\sigma_y}{\mu_p}$$

and

$$\alpha_{\lim}^{+} = 0.8\frac{\sigma_y}{p} = 0.8\frac{\sigma_y}{\mu_p\left(1+\kappa\nu_p\right)} = 0.8\frac{\sigma_y}{\mu_p\left(1+3.719\cdot 0.05\right)} = 0.675\frac{\sigma_y}{\mu_p}.$$

Tables 4.18, 4.19 show our results and from other authors obtained by different FEM discretization and approaches for equibiaxial tension ($p_1 \in [0,1],\ p_2 \in [0,1]$) and uniaxial tension ($p_1 \in [0,1],\ p_2 = 0$). Figures 4.26, 4.27 present the evolutions of limit and shakedown load factors under uniaxial tension. Figure 4.28 shows the interaction diagram of limit and shakedown analyses.

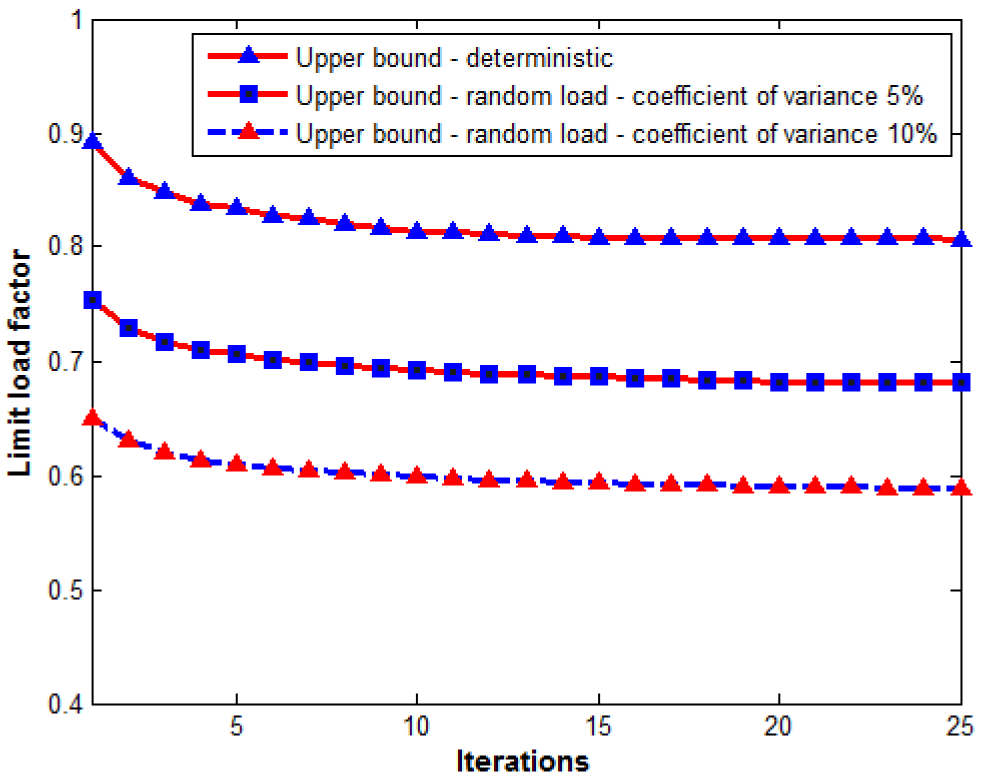


Figure 4. 26 Convergence of limit load factor in case of uniaxial loading

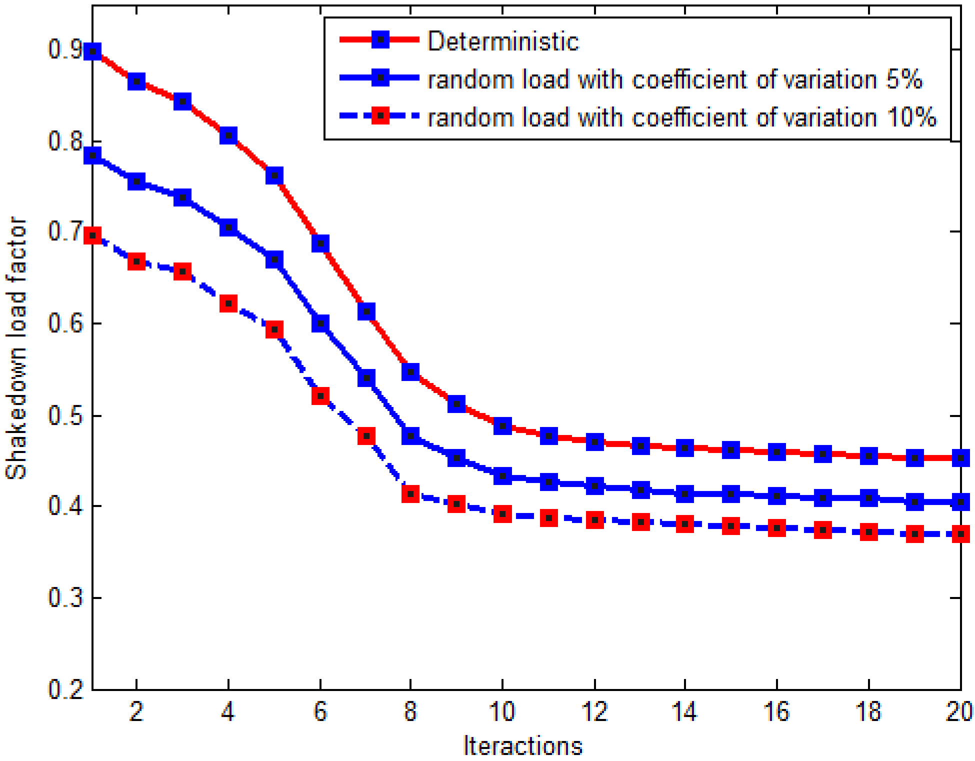


Figure 4. 27 Shakedown load factors with different coefficients of variation of load

Table 4. 18 Comparison of $\alpha^{+}_{\lim}$ with analytical solutions

	Present			
situations	Analytic $\nu = 0.05$	Upper bound $\nu = 0.05$	Analytic $\nu = 0.1$	Upper bound $\nu = 0.1$
Deterministic	0.8	0.802	0.8	0.802
Random R	0.651	0.657	0.502	0.507
Random L	0.675	0.685	0.583	0.596

Table 4. 19 Limit analysis: comparison with some authors

Tension loading	equibiaxial	uniaxial	
Belytschko [151]	—	0.780	deterministic
Corradi *et al.* [152]	0.767	0.691	
Genna [153]	—	0.793	
Groß-Weege [154]	0.882	0.782	
Garcea *et al.* [133]	0.902	0.806	
Tran *et al.* [26]	0.896	0.797	
Staat *and* Heitzer [117]	0.913		
Present	0.900	0.805	
	0.759	0.681	$\sigma_p = 0.05\mu_p$
		0.588	$\sigma_p = 0.1\mu_p$
Gaydon, McCrum [155]	—	0.800	deterministic
Present: analytical, exact	—	0.675	$\sigma_p = 0.05\mu_p$
	—	0.583	$\sigma_p = 0.1\mu_p$

Table 4. 20 Shakedown analysis: comparison with some authors

Tension loading	equibiaxial	uniaxial	
Belytschko [151]	0.431	0.571	deterministic
Corradi *et al.* [152]	0.504	0.654	
Genna [153]	0.478	0.653	
Groß-Weege [154]	0.446	0.614	
Garcea *et al.* [133]	0.438	0.604	
Tran *et al.* [26]	0.434	0.601	
Present	0.436	0.603	
	0.369	0.508	$\sigma_p = 0.05\mu_p$
	0.318	0.438	$\sigma_p = 0.1\mu_p$

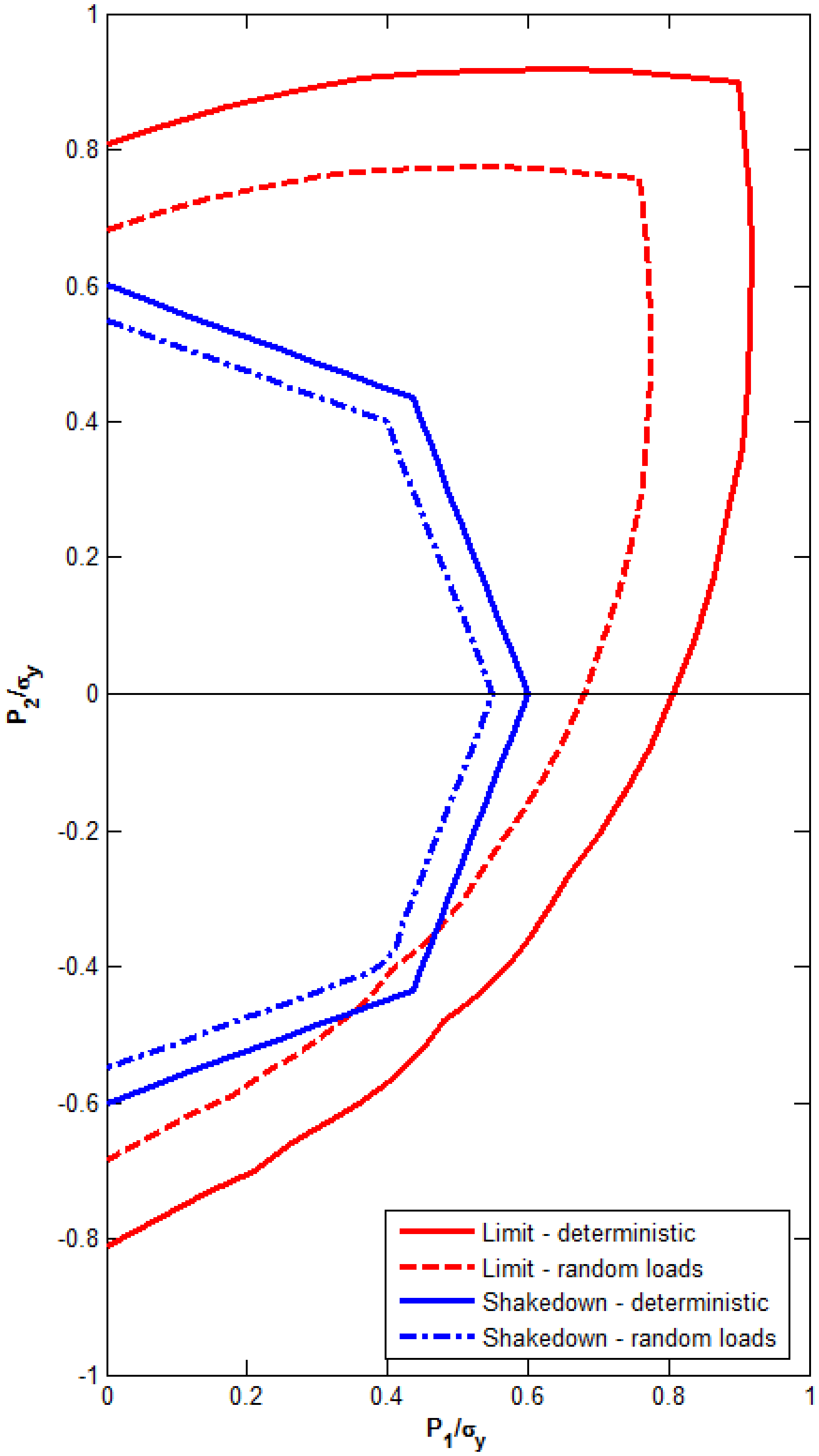


Figure 4. 28 Interaction diagram for the case of random load (deterministic strength)

- **Random strength and random loads**

  The analytic limit load for case of random load and random strength (coefficient of variation are 0.1 and 0.05 respectively) can be derived from (4.3).

  + If strength and load distributed normally:

$$\alpha_{\lim}^{+} = 0.8\frac{\sigma_y}{p} = 0.8\frac{\mu_R\left(1-\kappa\nu_R\right)}{\mu_p\left(1+\kappa\nu_p\right)} = 0.8\frac{\mu_R\left(1-3.719\cdot 0.1\right)}{\mu_p\left(1+3.719\cdot 0.1\right)} = 0.366\frac{\sigma_y}{\mu_p}$$

and

$$\alpha_{\lim}^{+} = 0.8\frac{\sigma_y}{p} = 0.8\frac{\mu_R\left(1-\kappa\nu_R\right)}{\mu_p\left(1+\kappa\nu_p\right)} = 0.8\frac{\mu_R\left(1-3.719\cdot 0.05\right)}{\mu_p\left(1+3.719\cdot 0.05\right)} = 0.549\frac{\sigma_y}{\mu_p}.$$

Table 4.21 shows our results obtained from analytical solutions and numerical solutions in the case of unxial tension

Table 4. 21 Comparison of $\alpha_{\lim}^{+}$ with analytical solutions ($\nu_R = \nu_p = \nu$)

	Present (normally distributed strength, normally distributed load)			
situations	Analytic $\nu = 0.05$	Upper bound $\nu = 0.05$	Analytic $\nu = 0.1$	Upper bound $\nu = 0.1$
Deterministic	0.8	0.802	0.8	0.802
Random R	0.651	0.657	0.502	0.507
Random L	0.675	0.685	0.583	0.596
Random R+L	0.549	0.553	0.366	0.369

Table 4. 22 Limit analysis: comparison with some authors

Tension loading	equibiaxial	uniaxial	
Belytschko [151]	—	0.780	
Corradi *et al.* [152]	0.767	0.691	
Genna [153]	—	0.793	
Groß-Weege [154]	0.882	0.782	deterministic
Garcea *et al.* [133]	0.902	0.806	
Tran *et al.* [26]	0.896	0.797	
Staat *et al.* [117]	0.913		
Present	0.900	0.805	
	0.759	0.681	$\sigma_p = 0.05\mu_p$
		0.588	$\sigma_p = 0.1\mu_p$
Gaydon, McCrum [155]	—	0.800	deterministic
Present:	—	0.553	$\sigma_p = 0.05\mu_p, \sigma_R = 0.05\mu_R$
analystical, exact	—	0.369	$\sigma_p = 0.1\mu_p, \sigma_R = 0.1\mu_R$

Table 4. 23 Shakedown analysis: comparison with some authors

Tension loading	equibiaxial	uniaxial	
Belytschko [151]	0.431	0.571	deterministic
Corradi *et al.* [152]	0.504	0.654	
Genna [153]	0.478	0.653	
Groß-Weege [154]	0.446	0.614	
Garcea *et al* [133]	0.438	0.604	
Tran *et al.* [26]	0.434	0.601	
Present	0.436	0.603	
	0.262	0.371	$\sigma_p = 0.05\mu_p, \sigma_R = 0.05\mu_R$
	0.205	0.285	$\sigma_p = 0.1\mu_p, \sigma_R = 0.1\mu_R$

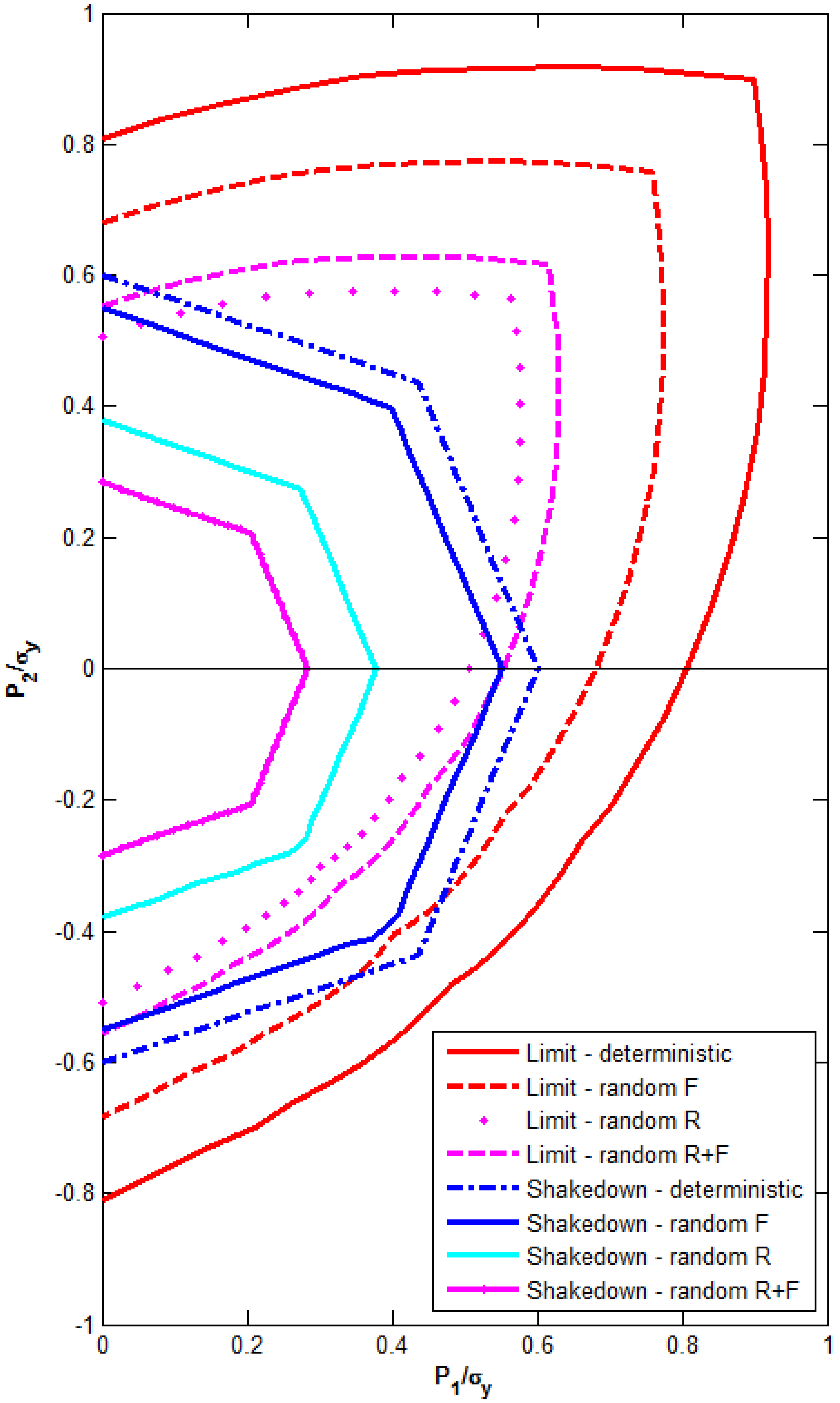


Figure 4. 29 Iteraction diagram with different situations of load and strength

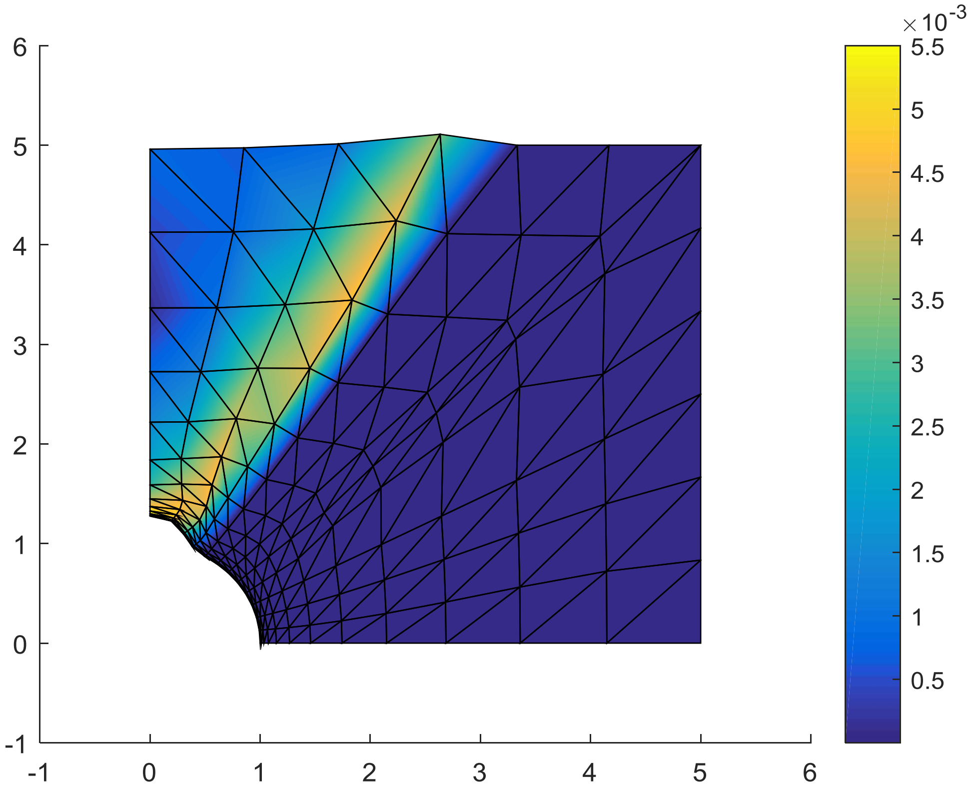


Figure 4. 30 Displacement rate plot showing the slip line in the plate with a hole in uniaxial tension

4.4. Shakedown analysis of a Kirchoff plate under uncertain strength

Plates are very important structural elements, which are widely used in civil and mechanical engineering. The common examples of plates are slabs in civil engineering structures, bearing plate under columns, many parts of mechanical components. In this chapter, we consider bending of such plates subjected to lateral loads. The bending properties of a plate depend greatly on its thickness. The classical theory divides plates into following groups:

- *Thin plates with small deflection*,
- *Thin plates with large deflections*, and
- *Thick plates*

In thin plates with small deflections theory, the following assumption are made:

a) *There is no deformation in the middle plane of the plate. This plane remains neutral during bending.*
b) *The normal to the middle plane of the plate remains straight and normal to the deformed middle plane.*
c) *The normal stresses in the direction transverse to the plate are negligible.*

The above assumptions were proposed by Gustav R. Kirchhoff on which A.E.H. Love based his plate theory. Consequently, thin plates with small deflections theory are called Kirchhoff-Love plate or Kirchhoff plate for short. This theory is suitable for plates with length of span is at least 10 times its thickness. Many engineering problems lie in the above category and satisfactory results are obtained by classical theories of thin plates.

If the span dimension is less than 10 times the thickness, the assumptions (a) and (b) listed under theory of thin plates will not hold good. Such plates need the Mindlin Theory of thick plates which accounts for shear deformations or a three dimensional analysis. The Mindlin theory developed for the analysis of such plates may be called thick plate theory.

Limit analysis of plates in bending has been studied analytically and numerically [51], [52], [156]–[163]. Due to limitations of analytical methods, alternative numerical approaches such as FEM methods, meshfree method, IGA method have been developed.

In [164] a dual algorithm has been developed to calculate simultaneously both the upper and lower bounds of the plastic collapse limit and shakedown limit of thin plate bending. This section, extents work based on the study of Tran [164] is considered in which the lower bound and upper bound limit and shakedown load of plate under uncertain strength will be computed.

4.4.1 Basic relations in thin plate theory

For the derivation of basic relations, one can refer to standard text books on analysis of plates [51,52]. In this section the necessary relations are listed taking the notations as indicated in the plate element shown in Fig. 4.31.

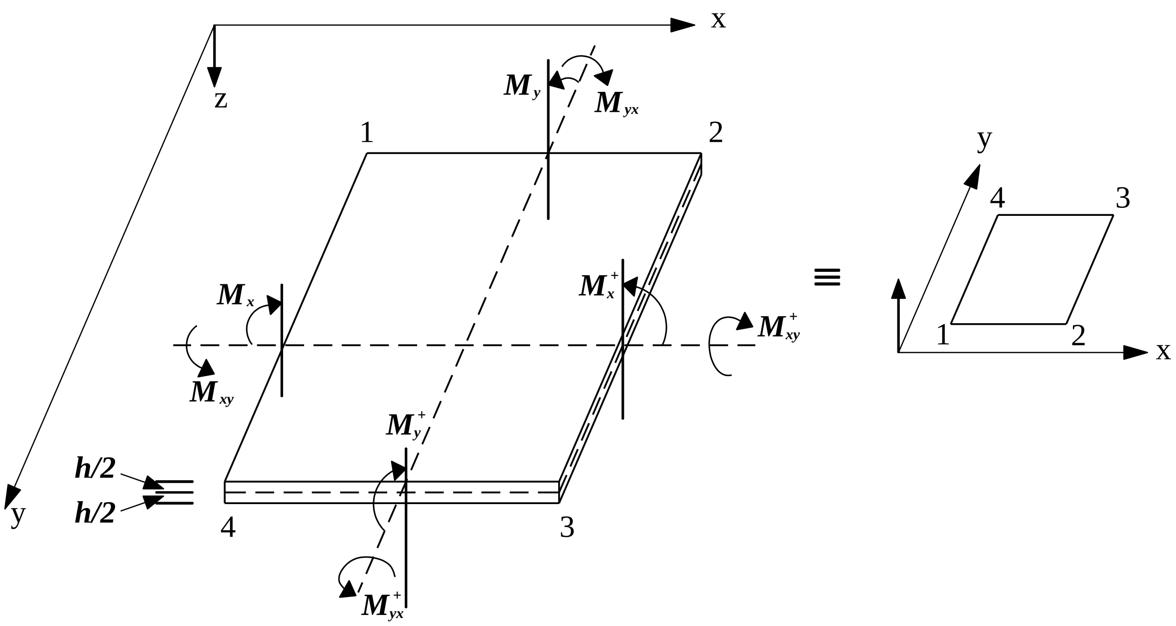


Figure 4. 31 Plate element with internal forces of bending moments

Let u, v and w be the displacement at any point (x, y, z) in the plate. Similar to the elastic theory, the inelastic behavior of thin plates may be analyzed under Kirchhoff's assumption that the normal to the middle plane of the plate remain straight and normal to the deformed middle plane. This assumption yields $u = -z\dfrac{\partial w}{\partial x},\ v = -z\dfrac{\partial w}{\partial y}$,

By differentiation, the strains are obtained as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \end{Bmatrix} = \begin{Bmatrix} -z\dfrac{\partial^2 w}{\partial x^2} \\ -z\dfrac{\partial^2 w}{\partial y^2} \\ -2z\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = z\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \tag{4.7}$$

The vector $\{\boldsymbol{\chi}\} = \left[\chi_x \ \chi_y \ \chi_{xy}\right]^T$ is called the vector of curvatures.

The kinematic relations can be written as follows:

$$\dot{\boldsymbol{\chi}} = -\nabla^2 \dot{w} \tag{4.8}$$

where $\dot{\boldsymbol{\chi}}$ is the curvature rate vector and $\dot{w}$ is the transversal velocity.

4.4.2 Yield criteria

Similar to bending of beams, the limit state of greatest capacity is obtained for a double rectangular distribution of the stresses across the depth of the plate. Therefore, the limit values of the bending moments in the x and y directions and of the twisting moment are

$$M_x = \sigma_x \frac{h^2}{4}, \ M_y = \sigma_y \frac{h^2}{4}, \ M_{xy} = \sigma_{xy} \frac{h^2}{4} \tag{4.9}$$

Contrary to the bending of beams, however, the normal stresses σ_x and σ_y are not equal to the yield limits in uniaxial tension; rather, they must satisfy a yield condition for plane stress, taking into account the in-plane shear stress σ_{xy}. The out-of-plane shear stresses, σ_{xz}, σ_{yz} are usually neglected. Consider a general yield condition of the form

$$f(\sigma_x, \sigma_y, \sigma_{xy}) = 0 \tag{4.10}$$

applicable to plane stress states. In a fully plasticized cross section, the stresses $\sigma_x, \sigma_y, \sigma_{xy}$ are constant. Expressing them from (4.6) in terms of bending and twisting moments, we have the corresponding yield criterion for a plate,

$$f\left(\frac{4M_x}{h^2}, \frac{4M_y}{h^2}, \frac{4M_{xy}}{h^2}\right) = 0 \tag{4.11}$$

As an illustration, the *von Mises yield criterion* can be written in the form

$$f(\sigma_x, \sigma_y, \sigma_{xy}) \equiv \sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2 - \sigma_0^2 = 0 \tag{4.12}$$

Expressing from (4.9) stresses in terms of moments, the criterion takes the form

$$f(M_x, M_y, M_{xy}) \equiv M_x^2 - M_xM_y + M_y^2 + 3M_{xy}^2 - M_0^2 = 0 \tag{4.13}$$

in which

$$M_0 = \sigma_0 \frac{h^2}{4} \tag{4.14}$$

In matrix form the von Mises yield criterion can be written as follows:

$$f(\mathbf{m}) = \sqrt{\mathbf{m}^{\mathrm{T}}\mathbf{P}\,\mathbf{m}} - m_0 = 0 \tag{4.15}$$

Where $\mathbf{m} = \{M_x, M_y, M_{xy}\}^{\mathrm{T}}$ is the vector of bending and twisting moments, $m_0 \equiv M_0$ is the plastic limit moment per unit length of a plate section and h is the thickness of plate, σ_0 is the uniaxial yield stress of material,

$$\mathbf{P} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{4.16}$$

If we consider the Tresca criterion, we have

$$f(\sigma_1, \sigma_2) \equiv \max\left(|\sigma_1|, |\sigma_2|, |\sigma_1 - \sigma_2|\right) - \sigma_0 = 0 \tag{4.17}$$

where σ_1, σ_2 are the principal stresses in the plane of plate. In term of the principal bending moments, M_1 and M_2, the Tresca criterion becomes

$$F(M_1, M_2) \equiv \max\left(|M_1|, |M_2|, |M_1 - M_2|\right) - M_0 = 0 \tag{4.18}$$

4.4.3 Static approach for deterministic problem

Consider a convex polyhedral load domain $\mathcal{L}$ and a special loading path consisting of all load vertices $\hat{P}_k\,(k = 1,\ldots,m)$ of $\mathcal{L}$. The total moment $\mathbf{m}(\mathbf{x},t)$ at a point $\mathbf{x} \in \Omega$ of the considered plate $\mathscr{P}$ at time t is decomposed into and elastic reference moment $\mathbf{m}^E(\mathbf{x},t)$ and a residual moment $\boldsymbol{\rho}(\mathbf{x},t)$. Here, $\mathbf{m}^E(\mathbf{x},t)$ denotes the fictitious moment that would appear in a purely elastic reference structure $\mathscr{P}^E$ under the same conditions as the original one, and $\boldsymbol{\rho}(\mathbf{x},t)$ represents residual moment field are induced by the evolution of plastic strains.

$$\mathbf{m}(\mathbf{x},t) = \mathbf{m}^E(\mathbf{x},t) + \boldsymbol{\rho}(\mathbf{x},t) \tag{4.19}$$

According to Melan's statically shakedown theorem, if there exists a time-independent residual moment field $\bar{\boldsymbol{\rho}}(\mathbf{x})$ such that the yield condition is satisfied for any loading path at any time t and in any point $\mathbf{x}$ of the plate, then the structure will shakedown. Based on this theorem, for a plate made up of elastic perfectly plastic material, the maximum enlarging of the load domain allowing still for shakedown, characterized by load factor α^- can be obtained by solving the following optimization problem:

$$\begin{aligned} &\alpha^- = \max \alpha \\ &\text{s.t.:}\begin{cases} \nabla^2 \bar{\boldsymbol{\rho}}(\mathbf{x}) = 0 & \text{in } \Omega \\ f\left[\alpha \mathbf{m}^{\mathrm{E}}(\mathbf{x},t) + \bar{\boldsymbol{\rho}}(\mathbf{x})\right] \le m_0 \end{cases} \end{aligned} \tag{4.20}$$

By discretizing the entire problem domain Ω into finite elements and applying the Gauss-Legendre integration technique, Eqs. (4.17) can be rewritten in the following form:

$$\begin{aligned} &\alpha^- = \max \alpha \\ &\text{s.t.:}\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\boldsymbol{\rho}}_i = 0 & \text{in } \Omega \\ f(\alpha \mathbf{m}_{ik}^E + \bar{\boldsymbol{\rho}}_i) \le m_0 & \forall i = \overline{1, NG} \quad \forall k = \overline{1, m} \end{cases} \end{aligned} \tag{4.21}$$

in which $\mathbf{B}_i$ is the deformation matrix, w_i is integration weight at Gauss point i ; NG denotes the total number of Gauss points of the structure.

4.4.4 Static approach to chance constrained programming

Consider the situation that the plastic moment of plate is not given but must be modelled $m_0 = m_0(\omega)$ through random variables on certain probability space. Under uncertainty, the inequalities of (7.15) are not always satisfied, the probability of the i^{th} yield condition being satisfied is greater than some reliability level ψ_i. Problem (4.21) becomes a chance constraint stochastic program:

$$\alpha^{-} = \max \alpha$$
$$s.t.: \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{0} \qquad \text{in } \Omega \\ \text{Prob}\left[f(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i) - m_{0i}(\omega) \le 0 \right] \ge \psi \qquad \forall i = \overline{1, NG} \qquad \forall k = \overline{1, m} \end{cases} \tag{4.22}$$

Let us consider the i^{th} individual chance constraint ($m_{0i}(\omega)$ is replaced by $m_i(\omega)$ for simplicity)

$$\text{Prob}\left[f\left(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right) - m_i(\omega) \le 0 \right] = \text{Prob}\left[f_i - m_i(\omega) \le 0 \right] \ge \psi_i \tag{4.23}$$

and assume that the plastic moment $m_i(\omega)$ is distributed normally with mean value μ_i and standard deviation σ_i. This means that $\ln\left[r_i(\omega)\right]$ is distributed normally with mean μ_i and standard deviation σ_i, in short $m_i \sim \mathrm{N}(\mu_i, \sigma_i^2)$.

Let us transform to standard normal distribution. The yield condition in (4.22) can be written as $\dfrac{f_i - \mu_i}{\sigma_i} \le \dfrac{m_i(\omega) - \mu_i}{\sigma_i}$ and we have:

$$\text{Prob}\left[f_i \le m_i(\omega) \right] = \text{Prob}\left[\frac{f_i - \mu_i}{\sigma_i} \le \frac{m_i(\omega) - \mu_i}{\sigma_i} \right] \tag{4.24}$$

Using the property of the cumulative distribution function (c.d.f.) of the standard normal distribution $\Phi(-x) = 1 - \Phi(x)$, we can write (4.24)as follows:

$$\text{Prob}\left[\frac{f_i - \mu_i}{\sigma_i} \le \frac{m_i(\omega) - \mu_i}{\sigma_i} \right] = 1 - \Phi\left[\frac{f_i - \mu_i}{\sigma_i} \right] = \Phi\left[\frac{\mu_i - f_i}{\sigma_i} \right]. \tag{4.25}$$

Now the probabilistic condition (4.23) is replaced by

$$\Phi\left[\frac{\mu_i - f_i}{\sigma_i} \right] \ge \psi_i \tag{4.26}$$

Introducing a new variable $\kappa_i = \Phi^{-1}(\psi_i)$ so that $\psi_i = \Phi(\kappa_i)$, inequality (4.26) becomes:

$$\Phi\left[\frac{\mu_i - f_i}{\sigma_i}\right] \geq \Phi(\kappa_i)\,. \tag{4.27}$$

Because Φ is monotonic it holds

$$\kappa_i \leq \frac{\mu_i - f_i}{\sigma_i} \qquad \text{or} \qquad f_i \leq \mu_i - \kappa_i \sigma_i\,. \tag{4.28}$$

Finally we get an equivalent deterministic formulation of the static approach:

$$\begin{aligned} &\alpha^- = \max \alpha \\ &\text{s.t.:}\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = 0 & \text{in } \Omega \\ f(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i) \leq \mu_i - \kappa\sigma_i & \forall i = \overline{1, NG} \quad \forall k = \overline{1, m} \end{cases} \end{aligned} \tag{4.29}$$

If plastic moment distributed lognormally with parameter μ_i, σ_i , by the similar way, the equivalent deterministic formulation can be obtained as

$$\begin{aligned} &\alpha^- = \max \alpha \\ &\text{s.t.:}\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = 0 & \text{in } \Omega \\ f(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i) \leq e^{\mu_i - \kappa\sigma_i} & \forall i = \overline{1, NG} \quad \forall k = \overline{1, m} \end{cases} \end{aligned} \tag{4.30}$$

In general, we can write problems (4.29) and problem (4.30) in one form as

$$\begin{aligned} &\alpha^- = \max \alpha \\ &\text{s.t.:}\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\rho}_i = 0 & \text{in } \Omega \\ f(\alpha \mathbf{m}_{ik}^E + \overline{\rho}_i) \leq \overline{m}_i & \forall i = \overline{1, NG} \quad \forall k = \overline{1, m} \end{cases} \end{aligned} \tag{4.31}$$

where:

$\overline{m}_i = \mu_i - \kappa\sigma_i$ for normal distribution of plastic moment

$\overline{m}_i = e^{\mu_i - \kappa\sigma_i}$ for lognormal distribution of plastic moment

4.4.6 Kinematic formulation for deterministic problem

An upper bound to the shakedown limit of plate can be obtained using the kinematic shakedown theorem which has following two statements:

Shakedown will occur for a structure subject to repeated or cyclic loads, if the rate of plastic dissipation power exceeds the work rate of external forces for any admissible plastic strain-rate cycles and all loading paths

Shakedown cannot occur, if the rate of plastic dissipation power is less than the work rate of external forces for any one admissible plastic strain-rate cycle or any one loading path.

In this investigation, we use von Mises yield criterion. The power of plastic dissipation per unit area of plate can be formulated as a function of strain rate:

$$\dot{D}_p = \sigma_0 \sqrt{\dot{\boldsymbol{\varepsilon}}^T \mathbf{Q}\, \dot{\boldsymbol{\varepsilon}}} \tag{4.32}$$

where

$$\mathbf{Q} = \mathbf{P}^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.33}$$

The plastic dissipation power of the plate domain Ω can be written

$$\dot{D}_{\mathrm{int}}(\dot{\boldsymbol{\chi}}) = \int_{\Omega} \int_{-h/2}^{h/2} \dot{D}_p \,\mathrm{d}z\,\mathrm{d}\Omega = m_0 \int_{\Omega} \sqrt{\dot{\boldsymbol{\chi}}^T \mathbf{Q}\,\dot{\boldsymbol{\chi}}}\,\mathrm{d}\Omega \tag{4.34}$$

in which m_0 is the plastic limit moment per unit length of a plate section is computed as (4.14)

We introduce here an admissible cycle of plastic curvature field $\Delta\boldsymbol{\chi}^p$. At each load vertex, the plastic curvature rate may not necessarily be compatible at each instant during the time cycle, but the plastic curvature accumulation over the cycle is required to be kinematic compatible such that

$$\Delta\boldsymbol{\chi}^p = \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}^p = \nabla^2 \dot{w} \tag{4.35}$$

Based on the above statements and the mathematical programming theory, an upper bound of the shakedown load factor can be found by solving the following convex nonlinear programming (the superscript p is neglected for simplicity):

$$\begin{aligned} &\alpha^+ = \min \sum_{k=1}^{m} \int_A \dot{D}_{\mathrm{int}}(\dot{\boldsymbol{\chi}})\,\mathrm{d}\Omega \\ &\text{s.t.:}\begin{cases} \Delta\boldsymbol{\chi}^p = \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}^p = \nabla^2 \dot{w} & \text{in } \Omega \\ \dot{w} = 0 & on \ \ \partial\Omega \\ \sum_{k=1}^{m} \int_{\Omega} \mathbf{m}^E(x, \hat{P}_k)\dot{\boldsymbol{\chi}}^T \mathrm{d}\Omega = 1 \end{cases} \end{aligned} \tag{4.36}$$

We denote the nodal variables of the finite element by $\mathbf{u} = [w \quad \partial w/\partial x \quad \partial w/\partial y]^{\mathrm{T}}$, the discretized formulation by FEM is as follows:

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0 \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{4.37}$$

4.4.7 Kinematic approach to chance constrained programming

As mentioned above the deterministic formulation to calculate a upper bound shakedown load factor:

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0 \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{4.38}$$

If the yield stress of material is random, then the plastic moment is uncertainty quantity and the objective function of (4.38) is a stochastic variable. Firstly, we must properly define the minimum of a random function. This can be done in such a way that one looks for a minimum lower bound η objective function under the constraint that the probability of violation of that bound is prescribed [166]

$$\min \eta$$
$$\text{s.t.:} \begin{cases} \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0(\omega) \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}} \geq \eta \right) = \psi \\ \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{4.39}$$

Problem (4.39) is a stochastic program, it can be converted into an equivalent deterministic program. Let make some transformations,

Denote

$$\Lambda = \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0(\omega) \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}} \tag{4.40}$$

The first constraint can be written as

$$\mathrm{Prob}(\Lambda \geq \eta) = 1 - \mathrm{Prob}(\Lambda \leq \eta) = \psi$$

$$1 - \mathrm{Prob}\left(\frac{\Lambda - \mu_\Lambda}{\sigma_\Lambda} \leq \frac{\eta - \mu_\Lambda}{\sigma_\Lambda}\right) = \psi$$

where

$\dfrac{\Lambda - \mu_\Lambda}{\sigma_\Lambda}$ is a standardized stochastic variable

For the Gaussian distribution, we have

$$\mathrm{Prob}\left(\frac{\Lambda - \mu_\Lambda}{\sigma_\Lambda} \leq \frac{\eta - \mu_\Lambda}{\sigma_\Lambda}\right) = \Phi\left(\frac{\eta - \mu_\Lambda}{\sigma_\Lambda}\right)$$

Equation of probability $\mathrm{Prob}(\Lambda \leq \eta) = 1 - \psi$ equalizes to

$$\psi = 1 - \Phi\left(\frac{\eta - \mu_\Lambda}{\sigma_\Lambda}\right) = \Phi\left(\frac{\mu_\Lambda - \eta}{\sigma_\Lambda}\right)$$

Set $\psi = \Phi(\kappa)$ we have $\Phi^{-1}(\kappa) = \kappa = \dfrac{\mu_\Lambda - \eta}{\sigma_\Lambda}$ or $\mu_\Lambda - \kappa\sigma_\Lambda = \eta$

The separate chance constrained program has the deterministic equivalent:

$$\begin{aligned} &\min \eta \\ &\text{s.t.:}\begin{cases} \mu_\Lambda - \kappa\sigma_\Lambda = \eta \\ \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \end{aligned} \tag{4.41}$$

or

$$\begin{aligned} &\min\left(\mu_\Lambda - \kappa\sigma_\Lambda\right) \\ &\text{s.t.:}\begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \end{aligned} \tag{4.42}$$

Finally, we can write clearly the discretized upper bound of shakedown limit load to chance constrained programming:

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \left(\mu_i - \kappa\sigma_i \right) \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{4.43}$$

If the random strength $m_{0,i}(\omega)$ is distributed lognormally no exact distribution of the sum

$$\sum_{k=1}^{m} \sum_{i=1}^{NG} w_i m_0 \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}} \tag{4.44}$$

However, the lognormal central limit theorem guarantees that the sum of n independent positive random variables tends to lognormal as n grows large. The plastic dissipation in (4.38) and thus the transformation of (4.39) into an equivalent deterministic form can only be obtained as an approximation. Moreover, there is a duality between lower bound and upper bound formulations. Consequently, one can assume the deterministic equivalent of (4.39) as (4.41)

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i e^{(\mu_i - \kappa\sigma_i)} \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{4.45}$$

We also write problem (4.43) and (4.45) in one form as

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \bar{m}_i \sqrt{\dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\chi}}_{ik}}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\boldsymbol{\chi}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} & \forall i = \overline{1, NG} \\ \sum_{k=1}^{m} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\chi}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{4.46}$$

where:

$\bar{m}_i = \mu_i - \kappa\sigma_i$ for normal distribution of plastic moment

$\bar{m}_i = e^{\mu_i - \kappa\sigma_i}$ for lognormal distribution of plastic moment

4.4.8 Dual approach to chance constrained programming

For the sake of simplicity, we set some new notations:

$$\dot{\mathbf{k}}_{ik} = w_i \mathbf{Q}^{1/2} \dot{\boldsymbol{\chi}}_{ik} , \quad \mathbf{t}_{ik} = \left(\mathbf{Q}^{-1/2}\right)^T \mathbf{m}_{ik}^E , \quad \hat{\mathbf{B}}_i = w_i \mathbf{Q}^{1/2} \mathbf{B}_i \tag{4.47}$$

where

$$\mathbf{Q}^{1/2}\mathbf{Q}^{-1/2} = \mathbf{I}, \quad \mathbf{Q} = \left(\mathbf{Q}^{1/2}\right)^T \mathbf{Q}^{1/2} \tag{4.48}$$

By substituting (4.47) into (4.46) one obtains a simplified version for upper bound of shakedown limit load (primal problem)

$$\alpha^+ = \min \sum_{k=1}^{m} \sum_{i=1}^{NG} \bar{m}_i \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}$$

$$\text{s.t.}: \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall\, i = \overline{1, NG} \qquad \text{(a)} \\ \sum_{i=1}^{NG} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 & \qquad \text{(b)} \end{cases} \tag{4.49}$$

Starting from problem (4.49) as primal problem the minimum problem (4.46) with restrictions is transformed into an unrestricted problem by the Lagrange function $L = L(\dot{\mathbf{k}}_{ik}, \dot{\mathbf{u}}, \alpha, \boldsymbol{\beta}_i)$. With the Lagrange factors $\alpha, \boldsymbol{\beta}_i$ it holds

$$L = \sum_{i=1}^{NG} \left\{ \sum_{k=1}^{m} \bar{m}_i \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik} + \varepsilon_0^2} + \boldsymbol{\beta}_i^T \left(\sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right) \right\} - \alpha \left(\sum_{i=1}^{NG} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik}^T \mathbf{t}_{ik} - 1 \right) \tag{4.50}$$

The Lagrange dual function (4.50) can be rewritten as:

$$L = \sum_{i=1}^{NG} \sum_{k=1}^{m} \left(\frac{\bar{m}_i \dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}} - \boldsymbol{\beta}_i - \alpha \mathbf{t}_{ik} \right)^T \dot{\mathbf{k}}_{ik} + \sum_{i=1}^{Ne} \boldsymbol{\beta}_i^T \hat{\mathbf{B}}_i \dot{\mathbf{u}} + \alpha \ . \tag{4.51}$$

Due to the existence of a dual solution having no duality gap the primal α^+, the function $\min\limits_{\dot{\mathbf{k}}_{ik}, \dot{\mathbf{q}}} L(\dot{\mathbf{k}}_{ik}, \dot{\mathbf{u}}, \alpha, \boldsymbol{\beta}_i)$ must have a finite value α . Therefore,

$$\left(\frac{\bar{m}_i \dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}} - \boldsymbol{\beta}_i - \alpha \mathbf{t}_{ik} \right)^T \dot{\mathbf{k}}_{ik} \geq 0 \tag{4.52}$$

must be satisfied in (4.51) because otherwise we always have $\min\limits_{\dot{\mathbf{k}}_{ik}, \dot{\mathbf{u}}} L(\dot{\mathbf{k}}_{ik}, \dot{\mathbf{u}}, \alpha, \boldsymbol{\beta}_i) \to -\infty$.

At the maximum the Lagrange function $L(\dot{\mathbf{k}}_{ik}, \dot{\mathbf{u}}, \alpha, \boldsymbol{\beta}_i)$ has a saddle point, so that the optimal value is the solution of

$$\max_{\alpha,\boldsymbol{\beta}_i}\ \min_{\mathbf{k}_{ik},\dot{\mathbf{q}}} L(\dot{\mathbf{k}}_{ik},\dot{\mathbf{u}},\alpha,\boldsymbol{\beta}_i) \tag{4.53}$$

The necessary conditions for the minimum are

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\mathbf{k}}_{ik}} &= -\boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik} + \bar{m}_i \frac{\partial}{\partial \dot{\mathbf{k}}_{ik}} \frac{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}} \\
&= -\boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik} + \bar{m}_i \left(\frac{2\dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}} - \frac{\left(\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}\right)^2 \dot{\mathbf{k}}_{ik}}{\left(\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}\right)^3} \right) \\
&= -\boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik} + \bar{m}_i \left(\frac{\dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}} \right) = \mathbf{0}
\end{aligned} \tag{4.54}$$

$$\frac{\partial L}{\partial \dot{\mathbf{u}}} = \boldsymbol{\beta}_i^T \hat{\mathbf{B}}_i = \mathbf{0} \tag{4.55}$$

Therefore, at the minimum

$$\left(\frac{\bar{m}_i \dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}} - \boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik} \right)^T \dot{\mathbf{k}}_{ik} = 0 \tag{4.56}$$

With (4.55) and (4.56) the dual objective function is

$$l(\alpha,\boldsymbol{\beta}_i) = \min_{\mathbf{k}_{ik},\dot{\mathbf{u}}} L = L(\dot{\mathbf{k}}_{ik},\dot{\mathbf{u}},\alpha,\boldsymbol{\beta}_i) = \alpha\ . \tag{4.57}$$

Furthermore, it is possible to prove that condition (4.56) at the minimum is also equivalent to following equality:

$$\left\|\boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik}\right\| = \bar{m}_i\ . \tag{4.58}$$

Indeed, we can always choose a strain rate $\dot{\mathbf{k}}_{ik}$ parallel to the vector $\boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik}$ and therefore

$$\begin{aligned}
&\left(\frac{\bar{m}_i \dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}} - \boldsymbol{\beta}_i - \alpha\,\mathbf{t}_{ik} \right)^T \dot{\mathbf{k}}_{ik} = \frac{\bar{m}_i \dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}} - (\boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik})\dot{\mathbf{k}}_{ik} = \\
&= \bar{m}_i \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}} - \left\|\boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik}\right\| \cdot \left\|\dot{\mathbf{k}}_{ik}\right\| = \bar{m}_i \left\|\dot{\mathbf{k}}_{ik}\right\| - \left\|\boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik}\right\| \cdot \left\|\dot{\mathbf{k}}_{ik}\right\| = 0
\end{aligned} \tag{4.59}$$

from which equation (4.56) follows. Similarly we get

$$\left\|\boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik}\right\| \le \bar{m}_i\ . \tag{4.60}$$

Now we can state the dual problem of (4.49):

$$\alpha^{-} = \max_{\alpha,\boldsymbol{\beta}_i} \alpha$$
$$\text{s.t.:} \quad \begin{cases} \left\| \boldsymbol{\beta}_i + \alpha \mathbf{t}_{ik} \right\| \le \overline{m}_i \\ \sum_{i=1}^{NG} \hat{\mathbf{B}}_i^T \boldsymbol{\beta}_i = \mathbf{0} \end{cases} \tag{4.61}$$

Problem (4.61) can be written as follows after some transformations:

$$\begin{aligned} \left\| \boldsymbol{\beta}_i + \mathbf{t}_{ik}\alpha \right\| &= \left\| \boldsymbol{\beta}_i + \left(\mathbf{Q}^{-1/2}\right)^T \mathbf{m}_{ik}^E \alpha \right\| = \left\| \left(\mathbf{Q}^{-1/2}\right)^T \left[\left(\mathbf{Q}^{1/2}\right)^T \boldsymbol{\beta}_i + \mathbf{m}_{ik}^E \alpha \right] \right\| \\ &= \left\{ \left[\left(\mathbf{Q}^{1/2}\right)^T \boldsymbol{\beta}_i + \mathbf{m}_{ik}^E \alpha \right]^T \mathbf{Q}^{-1/2} \left(\mathbf{Q}^{-1/2}\right)^T \left[\left(\mathbf{Q}^{1/2}\right)^T \boldsymbol{\beta}_i + \mathbf{m}_{ik}^E \alpha \right] \right\}^{1/2} \\ &= \left\{ \left[\left(\mathbf{Q}^{1/2}\right)^T \boldsymbol{\beta}_i + \mathbf{m}_{ik}^E \alpha \right]^T \mathbf{Q}^{-1} \left[\left(\mathbf{Q}^{1/2}\right)^T \boldsymbol{\beta}_i + \mathbf{m}_{ik}^E \alpha \right] \right\}^{1/2} \end{aligned} \tag{4.62}$$

Using (4.15) ,(4.33) and (4.62), the inequality in (4.61) now becomes:

$$f\left(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right) \le \overline{m}_i \tag{4.63}$$

with $\overline{\boldsymbol{\rho}}_i = \left(\mathbf{Q}^{1/2}\right)^T \boldsymbol{\beta}_i$.

For the second constraint of (4.61) we substitute $\hat{\mathbf{B}}_i$ *by* $\mathbf{B}_i$ *from (4.47)*:

$$\sum_{i=1}^{NG} \hat{\mathbf{B}}_i^T \boldsymbol{\beta}_i = \left(w_i \mathbf{Q}^{1/2} \mathbf{B}_i\right)^T \boldsymbol{\beta}_i = w_i \mathbf{B}_i^T \left(\mathbf{Q}^{1/2}\right)^T \boldsymbol{\beta}_i = w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{0} \tag{4.64}$$

As a result, problem (4.61) is equivalent to problem (4.31):

$$\alpha^{-} = \max \alpha$$
$$s.t.: \begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \overline{\boldsymbol{\rho}}_i = 0 \qquad \text{in } \Omega \\ f\left(\alpha \mathbf{m}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right) \le \overline{m}_i \qquad \forall i = \overline{1, NG} \qquad \forall k = \overline{1, m} \end{cases}$$

The vector $\left(\mathbf{Q}^{1/2}\right)^T \boldsymbol{\beta}_i$ can be interpreted as the time-independent residual bending moment vector: the value of this vector is calculated at each Gauss point, independently of load vertices or, in other word, independently of time. On the other hand, the Lagrange multiplier α represents the lower bound shakedown load factor. The problem (4.61) is identical with problem (4.31). This mean (4.31) and (4.49) are dual to each other. Moreover, problem (4.46) is the deterministic equivalent form of (4.39).

4.4.9 A dual algorithm for shakedown analysis of a Kirchhoff plate

In order to allow a direct non-linear, smooth optimization problem, a 'smooth regularization method' should be used for overcoming this barrier. For this purpose, a very small positive number, ε_0^2 will be added to $D_{\text{int}}(\dot{\mathbf{k}}_{ik})$. An efficient technique for large-scale optimization problems, which are successfully applied in ([167],[129]) is used. Using penalty method to eliminate the first constraint in (4.49) lead to a penalty function

$$F_p = \sum_{i=1}^{NG} \left\{ \sum_{k=1}^{m} \bar{m}_i \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik} + \varepsilon_0^2} + \frac{c}{2} \left(\sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right)^T \left(\sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right) \right\} \tag{4.65}$$

where c is a penalty parameter such that $c \gg 1$.

The corresponding Lagrange function of (4.65) is

$$L = F_p - \alpha \left(\sum_{i=1}^{NG} \sum_{k=1}^{m} \dot{\mathbf{k}}_{ik}^T \mathbf{t}_{ik} - 1 \right) \tag{4.66}$$

We denote

$$\boldsymbol{\beta}_i = -c \left(\sum_{k=1}^{m} \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right) \tag{4.67}$$

By employing Newton method to solve the KKT conditions of the Lagrange function (4.66) and after some manipulations, one gets the following system:

$$\mathbf{K}\, \mathrm{d}\dot{\mathbf{u}} = -\mathbf{K}\dot{\mathbf{u}} + \mathbf{f}_1 + \mathbf{f}_2 (\alpha + \mathrm{d}\alpha) \tag{4.68}$$

in which

$$\begin{aligned}
\mathbf{K} &= \sum_{i=1}^{NG} \hat{\mathbf{B}}_i^T \mathbf{E}_i^{-1} \hat{\mathbf{B}}_i \\
\mathbf{f}_1 &= -\sum_{i=1}^{NG} \hat{\mathbf{B}}_i^T \mathbf{E}_i^{-1} \sum_{k=1}^{m} \mathbf{M}_{ik}^{-1} (\boldsymbol{\beta}_i + \alpha \mathbf{t}_{ik}) \frac{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik}}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik} + \varepsilon_0^2}} \\
\mathbf{f}_2 &= \sum_{i=1}^{NG} \hat{\mathbf{B}}_i^T \mathbf{E}_i^{-1} \sum_{k}^{m} \mathbf{M}_{ik}^{-1} \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik} + \varepsilon_0^2}\, \mathbf{t}_{ik}
\end{aligned} \tag{4.69}$$

and

$$\begin{aligned}
\mathbf{M}_{ik} &= \bar{m}_i \mathbf{I} + (\boldsymbol{\beta}_i + \alpha \mathbf{t}_{ik}) \frac{\dot{\mathbf{k}}_{ik}^T}{\sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik} + \varepsilon_0^2}} \\
\mathbf{E}_i &= \frac{\mathbf{I}}{c} + \sum_{k}^{m} \mathbf{M}_{ik}^{-1} \sqrt{\dot{\mathbf{k}}_{ik}^T \dot{\mathbf{k}}_{ik} + \varepsilon_0^2}
\end{aligned} \tag{4.70}$$

The system (4.68) with the two last terms on the right-hand side may be interpreted as the linear system arising in purely elastic computations with the global stiffness matrix. The matrix $\mathbf{E}_i^{-1}$ plays the role of the elastic matrix. Solving this system by the same procedure as for the purely elastic calculation will ensure the displacement rate condition on boundary to be satisfied automatically. We have the incremental vectors of nodal variables $\dot{\mathbf{u}}$, curvature rate $\dot{\mathbf{k}}_{ik}$ and $\boldsymbol{\beta}_i$ as follows :

$$
\begin{aligned}
&d\dot{\mathbf{u}} = d\dot{\mathbf{u}}_1 + d\dot{\mathbf{u}}_2\left(\alpha + d\alpha\right) \\
&d\dot{\mathbf{k}}_{ik} = (d\dot{\mathbf{k}}_{ik})_1 + (d\dot{\mathbf{k}}_{ik})_2\left(\alpha + d\alpha\right) \\
&d\boldsymbol{\beta}_i = (d\boldsymbol{\beta}_i)_1 + (d\boldsymbol{\beta}_i)_2\left(\alpha + d\alpha\right)
\end{aligned}
\tag{4.71}
$$

where

$$
\begin{aligned}
&d\dot{\mathbf{u}}_1 = -\dot{\mathbf{u}} + \mathbf{K}^{-1}\mathbf{f}_1 \\
&d\dot{\mathbf{u}}_2 = \mathbf{K}^{-1}\mathbf{f}_2 \\
&\left(d\dot{\mathbf{k}}_{ik}\right)_1 = -\mathbf{M}_{ik}^{-1}\sqrt{\dot{\mathbf{k}}_{ik}^T\dot{\mathbf{k}}_{ik} + \varepsilon_0^2}\,\left(d\boldsymbol{\beta}_i\right)_1 - \mathbf{M}_{ik}^{-1}\left(\overline{m}_i\dot{\mathbf{k}}_{ik} + \sqrt{\dot{\mathbf{k}}_{ik}^T\dot{\mathbf{k}}_{ik} + \varepsilon_0^2}\,\boldsymbol{\beta}_i\right) \\
&\left(d\dot{\mathbf{k}}_{ik}\right)_2 = -\mathbf{M}_{ik}^{-1}\sqrt{\dot{\mathbf{k}}_{ik}^T\dot{\mathbf{k}}_{ik} + \varepsilon_0^2}\,\left(d\boldsymbol{\beta}_i\right)_2 - \mathbf{M}_{ik}^{-1}\sqrt{\dot{\mathbf{k}}_{ik}^T\dot{\mathbf{k}}_{ik} + \varepsilon_0^2}\,\mathbf{t}_{ik} \\
&\left(d\boldsymbol{\beta}_i\right)_1 = -\mathbf{E}_i^{-1}\sum_k^m \mathbf{M}_{ik}^{-1}\overline{m}_i\dot{\mathbf{k}}_{ik} - \mathbf{E}_i^{-1}\left[\hat{\mathbf{B}}_i d\dot{\mathbf{u}}_1 - \left(\sum_{k=1}^m \dot{\mathbf{k}}_{ik} - \hat{\mathbf{B}}_i\dot{\mathbf{u}}\right)\right] - \boldsymbol{\beta}_i \\
&\left(d\boldsymbol{\beta}_i\right)_2 = -\mathbf{E}_i^{-1}\hat{\mathbf{B}}_i d\dot{\mathbf{u}}_2 - \mathbf{E}_i^{-1}\sum_k^m \mathbf{M}_{ik}^{-1}\sqrt{\dot{\mathbf{k}}_{ik}^T\dot{\mathbf{k}}_{ik} + \varepsilon_0^2}\,\mathbf{t}_{ik}
\end{aligned}
\tag{4.72}
$$

and

$$
\left(\alpha + d\alpha\right) = \frac{1 - \sum_{i=1}^{NG}\sum_{k=1}^{m}\mathbf{t}_{ik}^T\left[\dot{\mathbf{k}}_{ik} + \left(d\dot{\mathbf{k}}_{ik}\right)_1\right]}{\sum_{i=1}^{NG}\sum_{k=1}^{m}\mathbf{t}_{ik}^T\left(d\dot{\mathbf{k}}_{ik}\right)_2}
\tag{4.73}
$$

The vectors $\mathrm{d}\dot{\mathbf{q}}, \mathrm{d}\dot{\mathbf{k}}_{ik}, \mathrm{d}\boldsymbol{\beta}_i$ and $\mathrm{d}\alpha$ are actually Newton directions which assure that a suitable step along them will lead to a decrease of the objective function of the primal problem (4.46) and to an increase of the objective function of the objective function of the dual problem (4.58). Based on (4.66-4.68) we can update the vectors of $\dot{\mathbf{q}}, \dot{\mathbf{k}}_{ik}$, $\boldsymbol{\beta}_i$ and α. Now the dual algorithm for limit and shakedown analysis of thin plate subjected bending is presented:

Algorithm A.3

Step 1. Initialize the displacement and strain rate vectors $\dot{\mathbf{u}}^0, \dot{\mathbf{k}}^0$ such that the normalized condition (4.46b) is satisfied:

$$
\sum_{i=1}^{Ne}\sum_{k=1}^{m}\mathbf{t}_{ik}^T\dot{\mathbf{e}}_{ik}^0 = 1
$$

Set vector $\boldsymbol{\beta}_i^0$ equal to zero vectors: $\boldsymbol{\beta}_i^0 = \mathbf{0}$

Step 2. Calculate $(d\dot{\mathbf{k}}_{ik})_1, (d\dot{\mathbf{k}}_{ik})_2$ at current value of $\dot{\mathbf{k}}_{ik}$ from (4.70), $(\alpha + d\alpha)$ from (4.71), Calculate $d\dot{\mathbf{k}}_{ik}$ from (4.66)

Step 3. Calculate $d\dot{\mathbf{u}}_1, d\dot{\mathbf{u}}_2$ from (4.70) and then calculate $d\dot{\mathbf{u}}$ from (4.69).

Step 4. Calculate a suitable *step-length* λ_k by solving a sub-problem

$$\min F_p\left(\dot{\mathbf{u}} + \lambda_k d\dot{\mathbf{u}}, \dot{\mathbf{k}}_{ik} + \lambda_k d\dot{\mathbf{k}}_{ik}\right)$$

Update displacement, strain rate α as:

$$\begin{aligned} \dot{\mathbf{u}} &= \dot{\mathbf{u}} + \lambda_k d\dot{\mathbf{u}} \\ \dot{\mathbf{e}} &= \dot{\mathbf{e}} + \lambda_k d\dot{\mathbf{e}} \end{aligned}$$

Step 5. Calculate $d\boldsymbol{\beta}_i$ from (4.69, 4.70). Find a *step-length* λ_s such that:

$$\begin{aligned} &\lambda_s \to \max \\ &s.t.: \\ &\left\|\boldsymbol{\beta}_i + \alpha\,\mathbf{t}_{ik} + \lambda_s(d\boldsymbol{\beta}_i + \mathbf{t}_{ik} d\alpha)\right\| \le \overline{m}_i \end{aligned}$$

Update $\alpha, \boldsymbol{\beta}_i$:

$$\begin{aligned} \alpha &= \alpha + d\alpha \\ \boldsymbol{\beta}_i &= \boldsymbol{\beta}_i + d\boldsymbol{\beta}_i \end{aligned}$$

Step 6. Check convergence criteria: if they are satisfied then stop the algorithm, otherwise repeat steps 2-5.

4.4.10 Numerical examples

The numerical solutions of two problems are presented to test the performance of the dual shakedown algorithm A4. The four-node discrete Kirchhoff quadrilateral (DKQ) plate element is applied for structural discretization. For all examples considered the following was assumed: length $L = 10m$, plate thickness $t = 0.1m$, the mean value of yield stress $E(\sigma_0) = 250\,\mathrm{MPa}$ and the standard deviation $\sigma = 0.1E(\sigma_0)$ .The reliability level is assumed $\psi = 0.9999$.

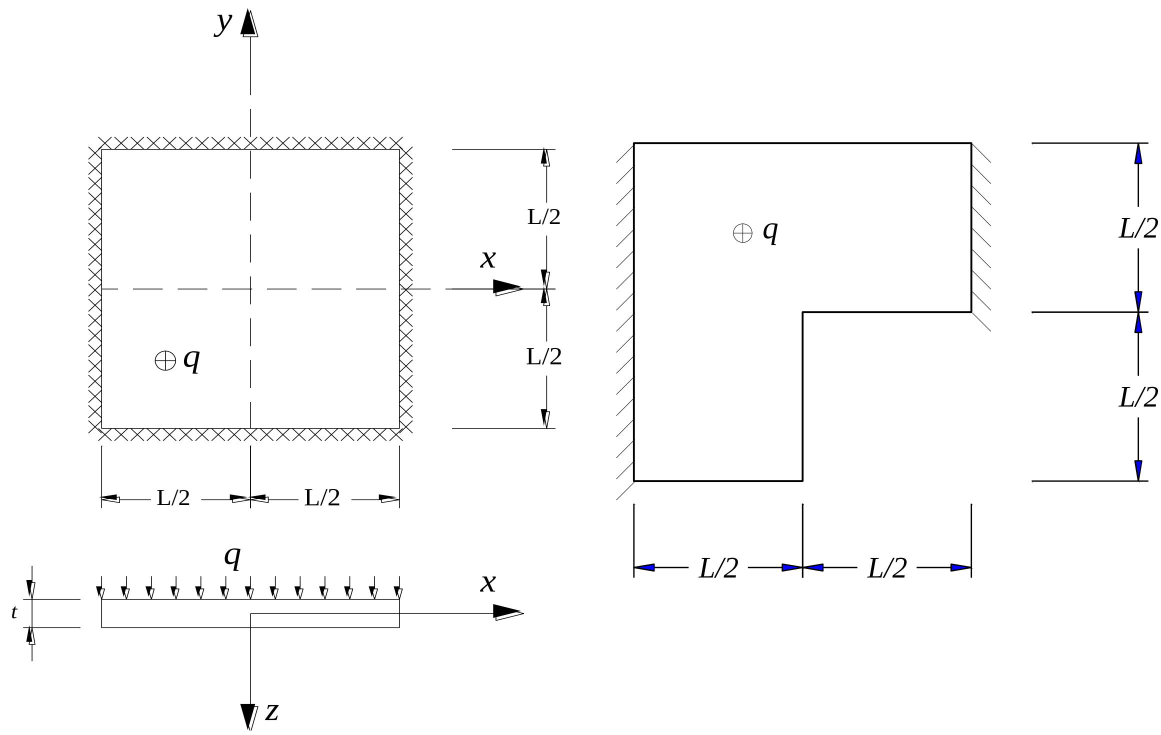


Figure 4. 32 Square plate and L-shape plate loaded by a uniform pressure

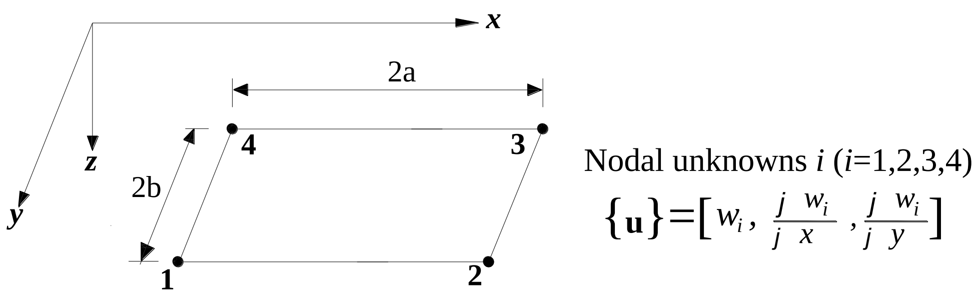


Figure 4. 33 Four-node rectangular plate bending element (DKQ element)

- **Example 1 : Square plate subjected to uniform load**

Firstly, we consider a square plate subjected to uniform pressure q as shown on Fig.4.30. In this analysis, the plate is modelled by 256DKQ elements due to symmetry. Tables 4.23-4.24 shows the comparison of present numerical results for both cases, simply supported and clamped plates. The results are normalized with m_0 / L^2 where L is the length of the plate. The exact solution for clamped plate was found by Fox [159] as $42.851\dfrac{m_0}{qL^2}$.

Figures 4.31, 4.32 show evolution of convergence of limit load factors. The advantage of the present method is that lower and upper bound are calculated simultaneously with no extra computational cost and they are coincident to ensure the accuracy of the solution.

Table 4. 24 Limit load factor in comparison for case of simple supported plate $\left(\dfrac{m_0}{qL^2}\right)$

Author	Lower bound	Upper bound	
Hodge *et al.* [156]	24.86	26.54	deterministic
Capsoni *et al.* [158]		25.02	
Lubliner [113]	23.81	27.71	
Le *et al.* [52]		25.01	
Tran *et al.* [164]	25.04	25.04	
Present	25.04	25.04	
	15.72	15.72	normal [167]
	17.19	17.19	lognormal

Table 4. 25 Limit load factor in comparison for case of clamped plate $\left(\dfrac{m_0}{qL^2}\right)$

Author	Lower bound	Upper bound	
Fox [159]	42.851	42.851	deterministic
Morley [157]		42.88	
Hodge *et al.* [156]	42.86	49.25	
Capsoni *et al.* [158]		45.29	
Lubliner [113]		52.01	
Le *et al.* [52]		45.29	
Tran *et al.* [164]	45.06	45.06	
Present	45.16	45.16	
	28.36	28.36	Normal [167]
	31.01	31.01	lognormal

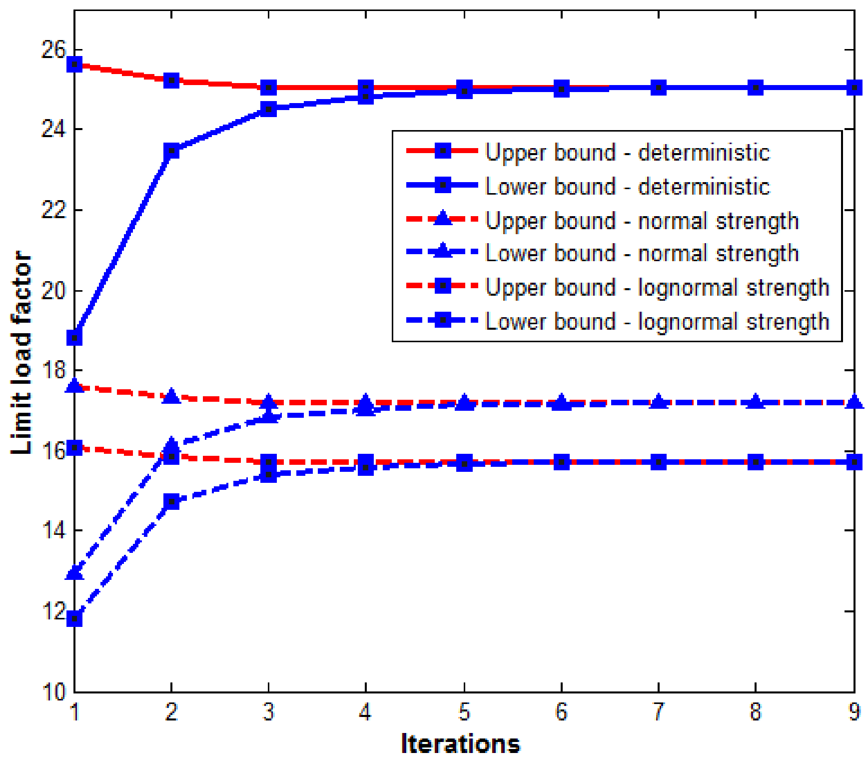


Figure 4. 34 Simple supported square plate: Convergence of limit load factors

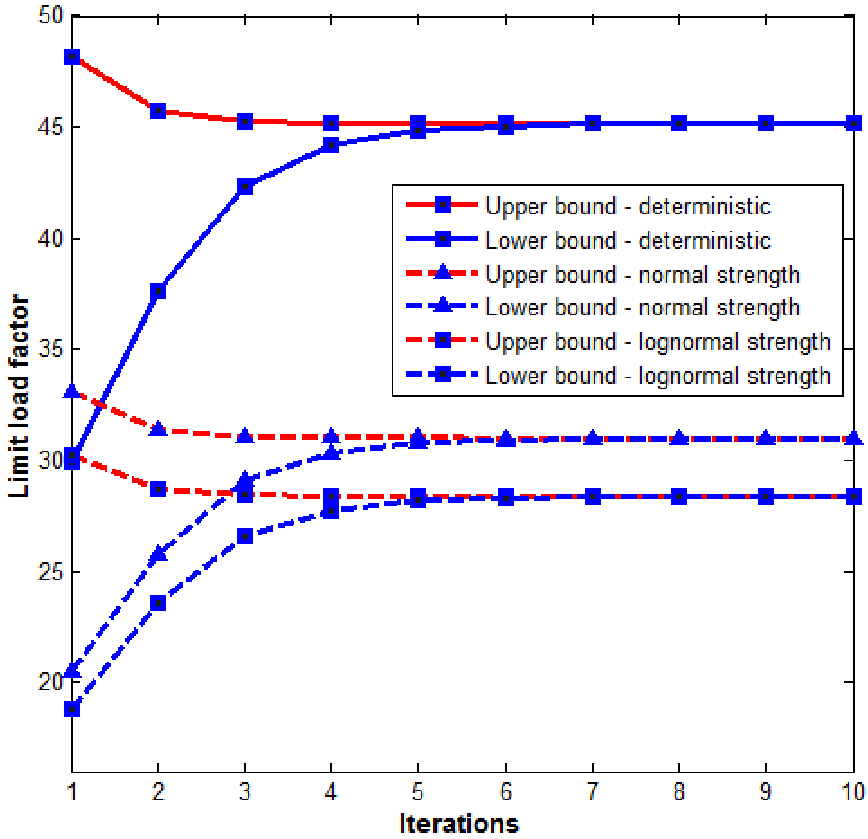


Figure 4. 35 Camped square plate: Convergence of limit load factors

Example 2: L-shaped plate subjected to uniform load.

In the second example, we investigate a L-shape plate subjected to uniform pressure q (Figure 4.29). This problem was studied in [51], [52] by upper bound limit for deterministic situation. Tran in [164] obtained deterministic solution for upper bound and lower bound limit load by a dual algorithm. In this analysis, the plate is modelled by 768 DKQ elements. Figure 4.33 shows the convergence of the upper bound and lower bounds for simple supported case. Table 4.25 shows the results in comparison with Le and Tran.

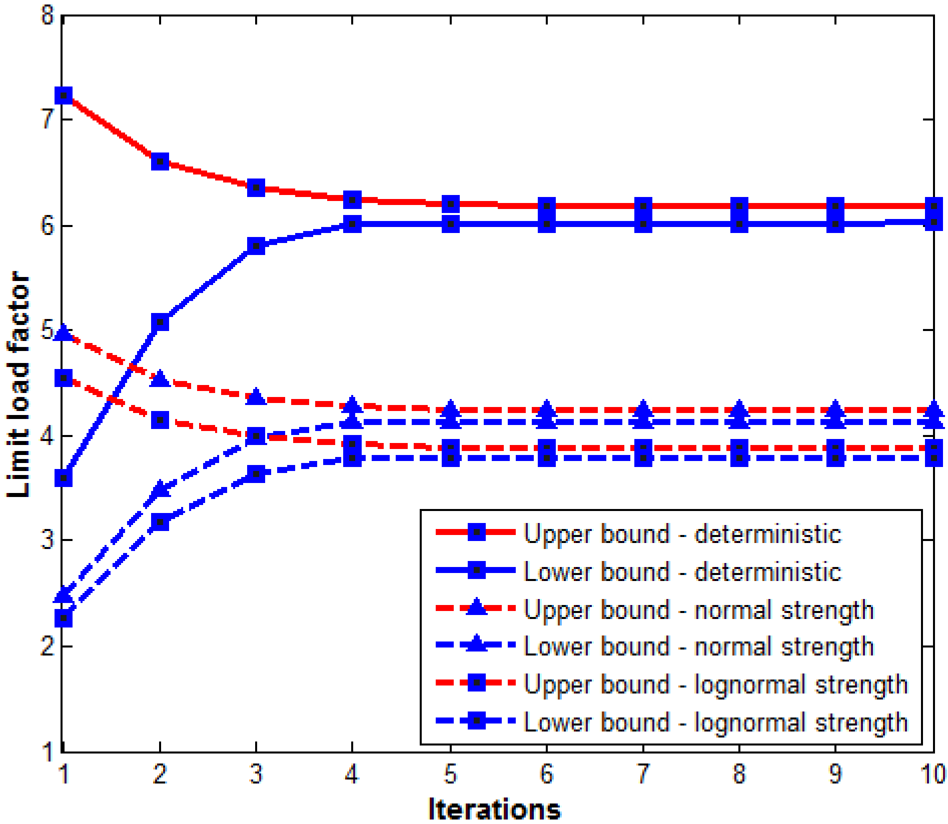


Figure 4. 36 L-shape Plate: Convergence of limit load factors

Table 4. 26 Limit load factor in comparison for case of simple supported plate $\left(\frac{m_0}{qL^2}\right)$

Author	Lower bound	Upper bound	
Le *et al.*	--	6.219	deterministic
Tran *et al.*	6.044	6.173	
Present	6.022	6.190	
	3.785	3.882	normal
	4.135	4.242	lognormal

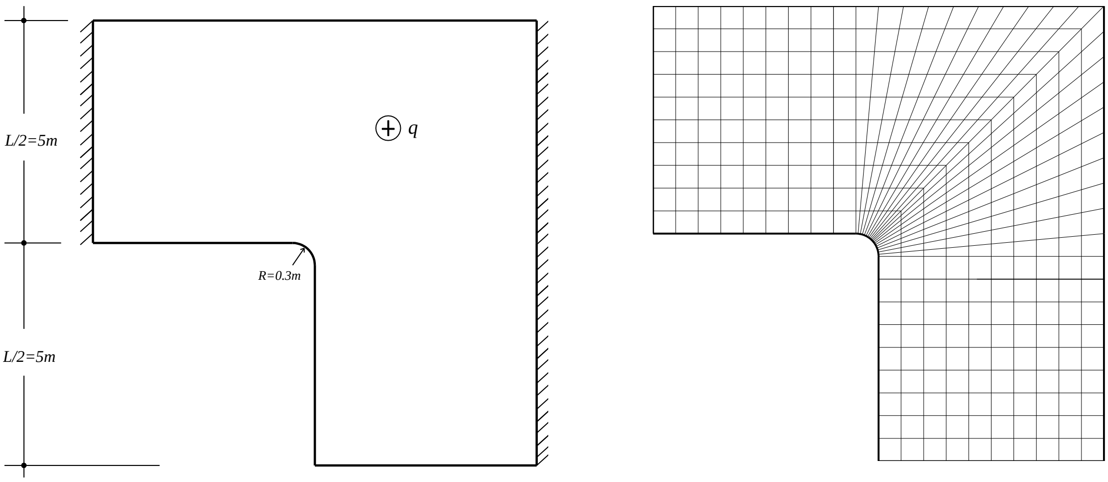


Figure 4.35: a. L-shaped plate with rounded corner b. FE mesh with 380 DKQ elements

Table 4.26 Limit and shakedown load factors for plate in Figure 4.35

	Lower bound	Upper bound	
Limit analysis	5.979	6.224	deterministic
	4.137	4.273	lognormal
	3.805	3.909	normal
Shakedown Analysis	5.339	5.355	deterministic
	3.619	3.677	lognormal
	3.363	3.382	normal

For shakedown analysis the stress singularity at the sharp renentrant corner has to be removed by rounding the plate at the corner as shown in figure 4.35. The FE mesh is made with 380 DKQ elements. Table 4.26 shows the limit and shakedown load factors if uniform load varies in domain $q = [0\ 1]$. If we compare with the result in table 4.25, the limit load factors are similar. The convergence of shakedown load factors is shown in figure 4.36

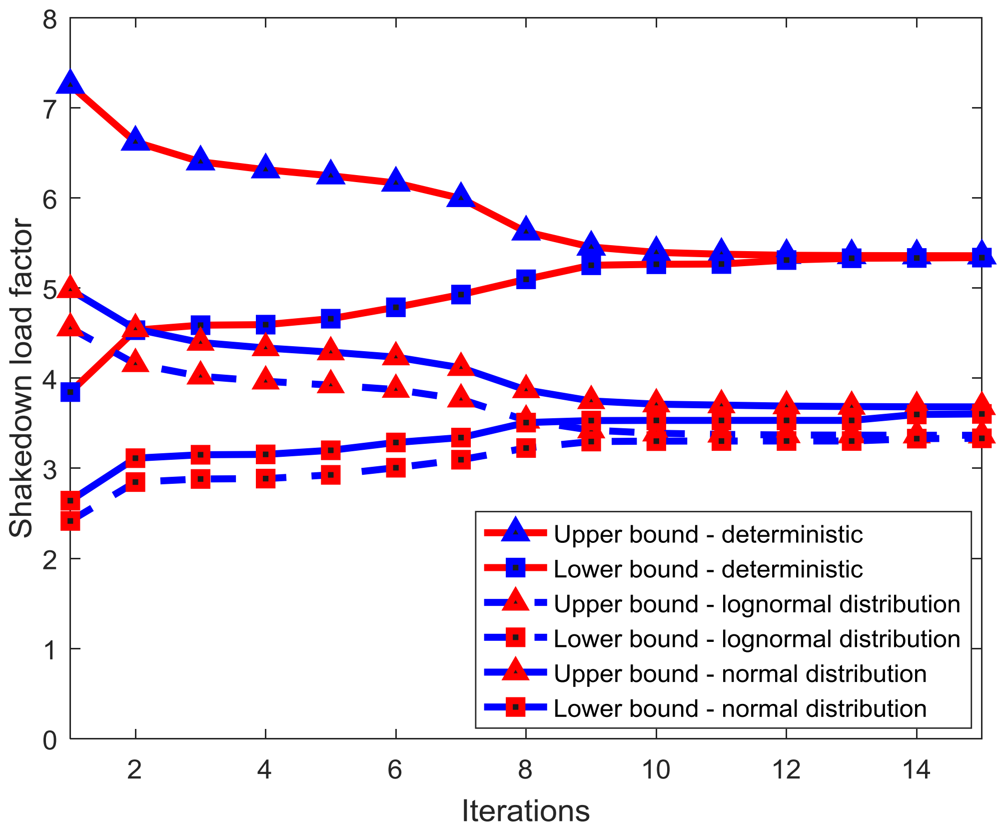


Figure 4.36 Convergence of shakedown load factors

Chapter 5

Reliability Analysis with the First Order Reliability Method

5.1 First Order Reliability Method

3.1.1 Introduction

In the chance constrained programming approach we have prescribed a reliability level and calculated the load factor. In structural reliability the failure probability is calculated for a given load factor. In order to find the relation between both approaches we consider briefly the First Order Reliability Method (FORM), which has been used in ([61], [63], [43], [130],[168]–[171],[191]) to calculate failure probabilities in limit and shakedown analysis. For more detail, see the given references.

Denote by $\mathbf{X} = (X_1, X_1, \ldots, X_n)$ an n-dimensional random vector characterizing uncertainties in the structure and load parameters. The limit state function $g(\mathbf{x}) = 0$, which is based on the comparison of a structural resistance (threshold) and loading, defines the limit state hyper-surface ∂V which separates the failure region $V = \{\mathbf{x} \,|\, g(\mathbf{x}) < 0\}$ from safe region.

$$g(\mathbf{X}) \begin{cases} < 0 & \text{for failure,} \\ = 0 & \text{for limit state,} \\ > 0 & \text{for safe structure.} \end{cases}$$

This is shown in Fig. 5.1 [43] after a transformation of $\mathbf{x}$ in the standard normal $\mathbf{u}$ space to be discussed below.

The failure probability P_f is the probability that $g(\mathbf{X})$ is non-positive, i.e.

$$P_f = P\left(g(\mathbf{X}) \le 0\right) = \int_F f_X(\mathbf{x})dx \tag{4.74}$$

where $f_X(\mathbf{x})$ is the n-dimensional joint probability density function. Usually, it is not possible to calculate P_f analytically. However, First- and Second-Order Reliability Methods (FORM/SORM) are analytical probability integration methods. Therefore, the defined problem has to fulfill the necessary analytical requirements (e.g. FORM/SORM apply to problems, where the set of basic variables are continuous. The numerical effort depends on the number of stochastic variables but not on P_f. Practical experience with FORM/SORM algorithms indicates that their estimates usually provide satisfactory reliability measures. Especially in the case of small failure probability (large reliability), FORM/SORM are extremely efficient compared with the MCS method regarding the requirement of computer time, such as the Central Processing Unit (CPU).

The failure probability is computed in three steps. Firstly the physical space $\mathbf{x}$ of uncertain parameters $\mathbf{X}$ is transformed into a new *n*-dimensional space $\mathbf{u}$ consisting of independent standard Gaussian variables $\mathbf{U}$. By this transformation, the original limit state $g(\mathbf{x})=0$ is mapped into the new limit state $g(\mathbf{u})=0$ in the $\mathbf{u}$ space. An expanded version of FORM in this notation can be found in [62].

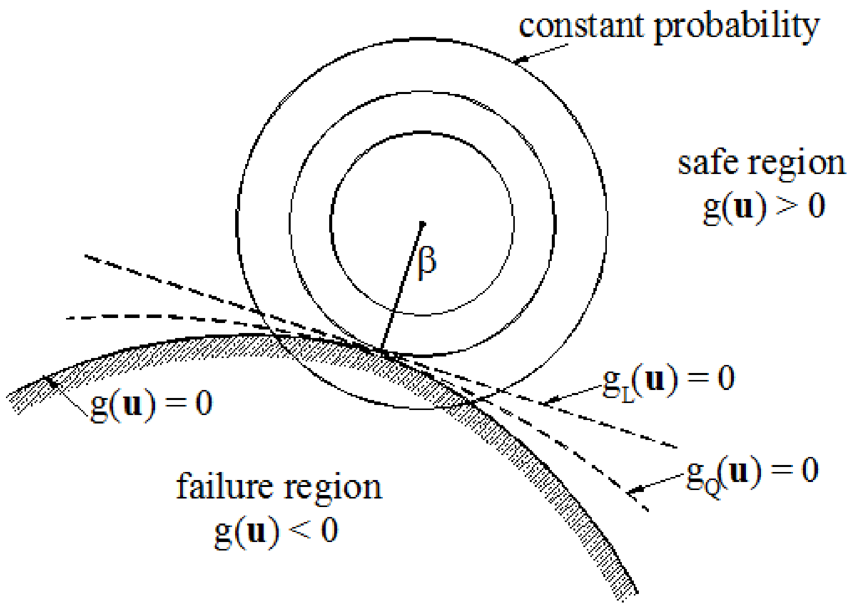


Figure 5. 1 Domains based on linear and quadratic approximations in $\mathbf{u}$ space [43]

In FORM, $g(\mathbf{U})=0$ with $g(\mathbf{0})>0$ is approximated linearly by its Taylor expansion $g_L(\mathbf{u})=g(\mathbf{u}^*)+\left(\nabla_u g(\mathbf{u}^*)\right)^T(\mathbf{u}-\mathbf{u}^*)$ at the so-called design point $\mathbf{u}^*\in\partial V$ (so that $g(\mathbf{u}^*)=0$)

$$g_L(\mathbf{u})=\beta+\boldsymbol{\alpha}^T\mathbf{u},$$

$$\boldsymbol{\alpha}=\frac{\nabla_u g(\mathbf{u}^*)}{\left\|\nabla_u g(\mathbf{u}^*)\right\|},\qquad \beta=-\boldsymbol{\alpha}^T\mathbf{u}^*.$$

The failure region V is linearly approximated by V_L

$$V_L=\left\{\mathbf{u}\,\middle|\,\beta+\boldsymbol{\alpha}^T\mathbf{u}\le 0\right\}=\left\{\mathbf{u}\,\middle|\,\boldsymbol{\alpha}^T\mathbf{u}\le-\beta\right\}.$$

The vector $\boldsymbol{\alpha}$ is proportional to the sensitivities $\nabla_u g(\mathbf{u}^*)$. The failure event $\{\mathbf{u}\in\partial V_L\}$ is equivalent to the event $\{\boldsymbol{\alpha}^T\mathbf{u}\le-\beta\}$, such that an approximation of the failure probability P_f is

$$P_f\approx P\left(\boldsymbol{\alpha}^T\mathbf{U}\le-\beta\right)=\Phi(-\beta)=\int_{-\infty}^{-\beta}e^{-0.5z^2}dz$$

because the random variable $\boldsymbol{\alpha}^T \mathbf{U}$ is normally distributed.

The failure probability depends only on the so-called called reliability (or safety) index β. For a linear limit state function FORM gives the exact failure probability $P_f = \Phi(-\beta)$. The limit state function is nearly linear for limit and shakedown analyses so that a quadratic second order approximation (SORM) $g_Q(\mathbf{u})$ of $g(\mathbf{u})$ is rarely needed.

5.2 Anlytical Reliability Analysis

An analytical FORM solution for the reliability index β and the failure probability P_f is known if X and Y in $\mathbf{X} = (X, Y)$ are both normally distributed or both lognormally distrubted. These solutions also hold if either X and Y is deterministic. This analytical FORM solution can be used to check the stochastic programming solutions and to understand the relation between the chance constraints and reliability analysis.

5.2.1 Reliability of the two-span continuous beam with normal distributions

With the analytical formulation of the upper bound (kinematic theorem) the limit state function is

$$g(x, y) = g\left(P_1, M_{0,1}\right) = 3M_{0,1} - \alpha P_1 \cdot L = 0 \,. \tag{4.75}$$

The normally distributed random variables $X = P_1$ and $Y = M_{0,1}$ with means $\mu_x = \mu_{P_1}$, $\mu_y = \mu_{M_{0,1}}$ and standard deviations $\sigma_x^2 = \sigma_{P_1}^2$, $\sigma_y^2 = \sigma_{M_{0,1}}^2$ are transformed in the standard normal variables $u_x = u_{P_1} = \dfrac{P_1 - \mu_{P_1}}{\sigma_{P_1}}$, $u_y = u_{M_{0,1}} = \dfrac{M_{0,1} - \mu_{M_{0,1}}}{\sigma_{M_{0,1}}}$. If $\mathbf{u} = \left(u_x, u_y\right)$ is $\mathrm{N}(0,1)$ distributed then $x = P_1 = \sigma_{P_1} u_{P_1} + \mu_{P_1}$, $y = \sigma_y u_y + \mu_y$ transforms the limit state function into

$$g\left(u_x, u_y\right) = 3\mu_{M_{0,1}} + 3\sigma_{M_{0,1}} u_{M_{0,1}} - \alpha L \mu_{P_1} - \alpha L \sigma_{P_1} u_{P_1} = 0 \,. \tag{4.76}$$

With realizations $\mathbf{u} = \left(u_x, u_y\right)$ of the new random variable $\mathbf{U}$ it may be written

$$g_L(\mathbf{u}) = \frac{3\sigma_{M_{0,1}} - \alpha L \sigma_{P_1}}{\sqrt{3^2 \sigma_{M_{0,1}}^2 + \alpha^2 L^2 \sigma_{P_1}^2}} \mathbf{u} + \frac{3\mu_{M_{0,1}} - \alpha L \mu_{P_1}}{\sqrt{3^2 \sigma_{M_{0,1}}^2 + \alpha^2 L^2 \sigma_{P_1}^2}} = \boldsymbol{\alpha}^T \mathbf{u} + \beta = 0 \tag{4.77}$$

such that the reliability index β is

$$\beta = \frac{3\mu_{M_{0,1}} - \alpha L \mu_{P_1}}{\sqrt{3^2 \sigma_{M_{0,1}}^2 + \alpha^2 L^2 \sigma_{P_1}^2}} \tag{4.78}$$

- **Random strength and deterministic load**

Let $M_{0,1}$ be normally distributed with the above mean value $\mu_{M_{0,1}}$ and standard deviation $\sigma_{M_{0,1}}$, respectively, and P_1 deterministic. The normally distributed random variable Y with mean with $y = \mu_{M_{0,1}} + \sigma_{M_{0,1}} u_{M_{0,1}}$ transforms the limit state function into

$$g\left(u_{M_{0,1}}\right) = 3\mu_{M_{0,1}} + 3\sigma_{M_{0,1}} u_{M_{0,1}} - \alpha P_1 \cdot L = 0 \,. \tag{4.79}$$

With realizations $\mathbf{u} = \left(u_{M_1}\right)$ of the new random variable $\mathbf{U}$ it may be written

$$g_L(\mathbf{u}) = \frac{3\sigma_{M_{0,1}}}{\sqrt{3^2 \sigma_{M_{0,1}}^2}} \mathbf{u} + \frac{3\mu_{M_{0,1}} - \alpha P_1 \cdot L}{\sqrt{3^2 \sigma_{M_{0,1}}^2}} = \boldsymbol{\alpha}^T \mathbf{u} + \beta \tag{4.80}$$

such that the reliability index β,

$$\beta = \frac{3\mu_{M_{0,1}} - \alpha P_1 \cdot L}{3\sigma_{M_{0,1}}} = \frac{3 \cdot 2\,\mathrm{kNm} - 1.256 \cdot 3\,\mathrm{kN} \cdot 1\mathrm{m}}{3 \cdot 0.2\,\mathrm{kNm}} = 3.72 \tag{4.81}$$

and the failure probability is $P_f = \Phi(-\beta) = \Phi(-3.72) = 1 \cdot 10^{-4}$. In this case comparing with the reliability $1 - P_f = \psi = \Phi(\kappa) = 0.9999$ we have

$$\kappa = \beta \tag{4.82}$$

- **Deterministic strength and random load**

Let $M_{0,1}$ be deterministic and P_1 normally distributed with the above mean value μ_{P_1} and standard deviation σ_{P_1}, respectively. The normally distributed random variable X with mean with $x = \mu_{P_1} + \sigma_{P_1} u_{P_1}$ transforms the limit state function into

$$g\left(u_x, u_y\right) = 3M_{0,1} - \alpha L \mu_{P_1} - \alpha L \sigma_{P_1} u_{P_1} = 0 \,. \tag{4.83}$$

With realizations $\mathbf{u} = \left(u_x, u_y\right)$ of the new random variable $\mathbf{U}$ it may be written

$$g_L(\mathbf{u}) = \frac{-\alpha L \sigma_{P_1}}{\sqrt{\alpha^2 L^2 \sigma_{P_1}^2}} \mathbf{u} + \frac{3M_{0,1} - \alpha L \mu_{P_1}}{\sqrt{\alpha^2 L^2 \sigma_{P_1}^2}} = \boldsymbol{\alpha}^T \mathbf{u} + \beta = 0 \tag{4.84}$$

such that the reliability index β is

$$\beta = \frac{3M_{0,1} - \alpha L \mu_{P_1}}{\alpha L \sigma_{P_1}} = \frac{3 \cdot 2 - 1.4578 \cdot 1 \cdot 3}{1.4578 \cdot 1 \cdot (0.1 \cdot 3)} = \frac{1.6266}{0.43734} = 3.719 \tag{4.85}$$

and the failure probability is $P_f = \Phi(-\beta) = \Phi(-3.719) = 1 \cdot 10^{-4}$. In this case we have the reliability $1 - P_f = \psi = \Phi(\kappa) = 0.9999$ and $\kappa = \beta = 3.719$.

5.2.2 Reliability of the two-span continuous beam with lognormal distributions

- **Random strength and deterministic load**

The transformation from X-space to U-space is nonlinear. The failure domain is given by

$$F = \left\{ (X,Y) \left| \frac{\alpha P_1 \cdot L}{3Y} \le 1 \right. \right\} = \left\{ (X,Y) \left| \ln\left(\frac{\alpha P_1 \cdot L}{3}\right) - \ln(Y) \le 0 \right. \right\} \tag{4.86}$$

With the limit state function

$$g(y) = g(M_{0,1}) = \ln(M_{0,1}) - \ln\left(\frac{\alpha P_1 \cdot L}{3}\right) = 0 \tag{4.87}$$

With the transformation we derive

$$g(y) = g(M_{0,1}) = u_y \sigma + \mu - \ln\left(\frac{\alpha P_1 \cdot L}{3}\right) = 0 \tag{4.88}$$

With realizations $\mathbf{u} = (u_y)$ of the new random variable $\mathbf{U}$ it may be written

$$g_L(\mathbf{u}) = \frac{\sigma}{\sqrt{\sigma^2}} \mathbf{u} + \frac{\mu - \ln\left(\frac{\alpha P_1 \cdot L}{3}\right)}{\sqrt{\sigma^2}} = \boldsymbol{\alpha}^T \mathbf{u} + \beta \tag{4.89}$$

such that the reliability index β is

$$\beta = \frac{\ln\left(\frac{\alpha P_1 \cdot L}{3}\right) - \mu}{\sigma} = \frac{0.6882\,\mathrm{kNm} - \ln\left(\frac{1.373 \cdot 3\,\mathrm{kN} \cdot 1\,\mathrm{m}}{3}\right)}{0.0998\,\mathrm{kNm}} = \frac{0.6882 - 0.317}{0.0998} = 3.719 \tag{4.90}$$

and the failure probability is $P_f = \Phi(-\beta) = \Phi(-3.719) = 1 \cdot 10^{-4}$.

5.2.4 Reliability of a square plate with a central hole with normal distributions

We can use the analysis for the case that strength and load are both normally distributed in [62] and the derive the two special cases with deterministic strength and deterministic load while the other variable is stochastic and normally distributed.

The limit load factor in eq. (4.3) for uniaxial loading $y = p_1$, $p_2 = 0$ and yield stress $x = \sigma_Y$ gives the limit state function

$$g(x, y) = \left(1 - \frac{D}{L}\right) x - y \tag{4.91}$$

The normally distributed random variables X and Y with means μ_x, μ_y and standard deviations σ_x^2, σ_y^2 are transformed in the standard normal variables u_x, u_y so that $x = \sigma_x u_x + \mu_x$, $y = \sigma_y u_y + \mu_y$. This transforms the limit state function into

$$G(u_x, u_y) = \left(1 - \frac{D}{L}\right)\mu_x - \mu_y + \left(1 - \frac{D}{L}\right)\sigma_x u_x - \sigma_y u_y \tag{4.92}$$

With realizations $\mathbf{u} = (u_x, u_y)$ of the standard normal variable $\mathbf{U}$ the limit state function can be written

$$G(\mathbf{u}) = \frac{\left(\left(1 - \frac{D}{L}\right)\sigma_x - \sigma_y\right)}{\sqrt{\left(1 - \frac{D}{L}\right)^2 \sigma_x^2 + \sigma_y^2}}\mathbf{u} + \frac{\left(1 - \frac{D}{L}\right)\mu_x - \mu_y}{\sqrt{\left(1 - \frac{D}{L}\right)^2 \sigma_x^2 + \sigma_y^2}} \tag{4.93}$$

so that the reliability index β is for $D/L = 0.2$:

$$\beta = \frac{\left(1 - \frac{D}{L}\right)\mu_x - \mu_y}{\sqrt{\left(1 - \frac{D}{L}\right)^2 \sigma_x^2 + \sigma_y^2}} = \frac{0.8\mu_x - \mu_y}{\sqrt{0.64\sigma_x^2 + \sigma_y^2}} \tag{4.94}$$

- **Random strength and deterministic load**

In the case of random strength with $\sigma_r = 0.1\mu_r$, limit load factor (from table 4.15) $\alpha_{\lim} = 0.50248$. Therefore following (4.94), the reliability index is computed as:

$$\beta = \frac{0.8\mu_x - \mu_y}{\sqrt{0.64\sigma_x^2 + \sigma_y^2}} = \frac{0.8\sigma_{yield} - 0.50248\sigma_{yield}}{\sqrt{0.64\left(0.1\sigma_{yield}\right)^2}} = \frac{0.29752}{0.8 \cdot 0.1} = 3.719 \tag{4.95}$$

In this case, $\sigma_y = 0$ because the load is deterministic.

- **Random load and deterministic strength**

In the case of deterministic strength, random load with $\sigma_p = 0.1\mu_p$, from table (4.20) the limit load factor $\alpha_{\lim} = 0.58313$. The reliability index β is taken from (4.94):

$$\beta = \frac{0.8\mu_x - \mu_y}{\sqrt{0.64\sigma_x^2 + \sigma_y^2}} = \frac{0.8\sigma_y - 0.58313\sigma_y}{\sqrt{0.64 \cdot 0^2 + \left(0.1 \cdot 0.58313\sigma_y\right)^2}} = \frac{0.21687}{0.058313} = 3.719 \qquad (4.96)$$

5.2.5 Reliability of a square plate with a central hole with lognormal distributions

If X and Y are lognormally distributed the random variables then the parameters $\mu_{x,y}, \sigma_{x,y}$

are calculated by (3.26).

The transformation from X-space to U-space is nonlinear. The failure domain is given by

$$F = \left\{ (X, Y) \left| \frac{(1 - D/L)X}{Y} \le 1 \right. \right\} = \left\{ (X, Y) \left| \ln\left(1 - D/L\right) + \ln(X) - \ln(Y) \le 0 \right. \right\} \qquad (4.97)$$

With the limit state function

$$g(X, Y) = \ln\left(1 - D/L\right) + \ln(X) - \ln(Y) \qquad (4.98)$$

With the transformation we derive

$$g(X, Y) = u_x\sigma_x - u_y\sigma_y + \ln(1 - D/L) + \mu_x - \mu_y \qquad (4.99)$$

such that the reliability index β is

$$\beta = \frac{\ln\left(1 - D/L\right) + \mu_x - \mu_y}{\sqrt{\sigma_x^2 + \sigma_y^2}} \qquad (4.100)$$

- **Lognormally distributed strength and deterministic load**

For this case, limit load factor from (4.6) is 0.5493

$$\beta = \frac{\ln\left(1 - D/L\right) + \mu_x - \mu_y}{\sqrt{\sigma_x^2 + \sigma_y^2}} = \frac{\ln(0.8) - 4.975 \cdot 10^{-3} - \ln(0.5493)}{9.975 \cdot 10^{-2}} = 3.719 \qquad (4.101)$$

5.2.6 The choice of safety factors on a probabilistic basis

The “unfired pressure vessel standard” prEN 13445-3 includes a design by analysis option, which uses limit and shakedown analysis. The considered limit states which can be based on limit and shakedown analysis are plastic collapse (gross plastic deformation, GPD), ratchetting (progressive deformation, PD), alternating plasticity (AP). Also other modern civil engineering design codes are based on structural limit states such as plastic collapse or buckling (instability, I). A second trend is that partial safety factors are defined for loading and resistance (strength) of the structure and attempts are made to base the partial safety factors on reliability analyses.

Table 5.1: Partial safety factors for GPD, prEN 13445-3. Partial safety factors for PD deformation are assumed to be unity because the failure develops slowly.

		Design check	
Actions	**Factor**	**DPD operation condition**	**GPD hydraulic testing**
Permamnent	γ_G		
unfavourable		1.2	1.2
favourable		0.8	0.8
Pressure	γ_p	1.2 (1.0)[#]	1.0
Variable	γ_Q	1.5 (1.0)[#]	–
Combination factor (stochastic actions)*	ψ	0.9	1.0
Resistance	γ_R	1.25	1.05
Temperature	γ_T	(1.0)	(1.0)

[#] With natural limit

* If not specified differently in the relevant code on environmental actions

We propose to use the deterministic equivalent approach to stochastic pogramming approach as alternative to safety factors. But even for deterministic partial safety factor the stochastic approach may give some hints. To this end we collect some analytical limit loads with the same normal distribution of resistance (R) and load (L). It shows that the partial safety factor for strength needs to be larger than the partial safety factor for loading for the same coefficient of variation $\nu = \sigma/\mu = 0.1$ and the same reliability level $\psi = 0.9999$ (i.e. failure probability $P_f = 1 - \psi = 1 \cdot 10^{-4}$ so that $\kappa = \Phi^{-1}(\psi) = \Phi^{-1}(0.9999) = 3.719$). The difference increases with ν .

Table 5.2: Comparison of analytic stochastic limit loads with $\sigma = 0.1\mu$, $P_f = 1 - \psi = 1 \cdot 10^{-4}$ for normally distributed R and L.

Distributions	**Two-span beam**		**Plate with central hole**	
	Load factor	**Safety factor**	**Load factor**	**Safety factor**
Deterministic	2	1	0.8	1
Random R	1.256	$\gamma_R = 1.592$	0.502	$\gamma_R = 1.594$
Random L	1.458	$\gamma_L = 1.372$	0.583	$\gamma_L = 1.372$
Random R+L	0.916	$\gamma_{R+L} = 2.183$	0.366	$\gamma_{R+L} = 2.186$

Let the safety factors in Table 5.2 be calculated with respect to the deterministic limit load factor $\alpha_{\det}$ so that $\alpha_R = \alpha_{\det}/\gamma_R$, $\alpha_L = \alpha_{\det}/\gamma_L$. Apparently, here

$\alpha_{R+L} = \alpha_{\det}/\gamma_{R+L} = \alpha_{\det}/(\gamma_R\gamma_L)$. In fact, the assumption $\gamma_{R+L} = \gamma_R\gamma_L$ in design codes seems to be justified.

The proposed chance constained programming approach to limit design has a great practical advantage over the structural reliability analysis which is performed after the structural design. It is well known that the relibalibility is strongly influenced by the tails of the distributions and the failure probability P_f can change over magnitudes for different stochastic models and small change of stochastic data. It has pointed out that the reliability index β is much less sensitive and the sensitivity is a consequence of the function $P_f = \Phi(-\beta)$ [61]. This good nature of β carries over to κ and as a consequence to the engineering decision of the designed limit or shakedown load factor. In fact, our examples show that a change of the order of the designed failure probability leads only to small changes of of few percents of the load factor (see Fig. 4.4). This makes the probabilistic design decisions relatively stable. Our analysis also shows that it is important to keep the uncertainty of strength and loading small because otherwise the uncertainty has to be paid by small load factors (see Fig. 4.3).

5.2.7 Relation between total reliability and partial reliabilities

For a given failure probability, the structure will fail corresponding to the minimum kinematic load factor $\alpha^+_{\lim}$ which is computed by solving problem (3.81):

$$\alpha^+_{\lim} = \min \eta$$

$$\text{s.t.:} \begin{cases} \operatorname{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}} \geq \eta \right) = \psi_r \\ \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} \\ \operatorname{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} < 1 \right) = \psi_{\mathrm{p}} \end{cases}$$

We have the following relations between the assumed probabilities ψ_r, ψ_p and the bounds $P_f^{total}, \psi^{total}$. Here P_f^{total} and ψ^{total} are the bounds of total failure probability and the total reliability of the structure, respectively.

The structure will fail if the external power $\alpha^+_{\lim} \dot{W}_{ext}(\omega)$ is larger than the plastic distipation $\dot{W}_{in}(\omega)$,

where

$$\dot{W}_{in}(\omega) = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i(\omega) \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}}, \qquad \dot{W}_{ext}(\omega) = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik}(\omega) \tag{4.102}$$

are stocatically independent because $r_i(\omega)$ and $\mathbf{t}_{ik}(\omega)$ are independent.

- **The strength is random and load is deterministic**

In section (3.2.1), the stochastic programm (3.49) reads

$$\alpha^+_{\lim} = \min \eta$$
$$\text{s.t.:} \begin{cases} \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}} \geq \eta \right) = \psi_r \\ \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} \quad = \mathbf{0} \\ \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 \end{cases}$$

Let us make some transformations:

$$P_f^{total} = 1 - \psi^{total} = \text{Prob}\left(\dot{W}_{in}(\omega) < \alpha^+_{\lim} \dot{W}_{in}(\omega)_{ext} \right) = \text{Prob}\left(\dot{W}_{in}(\omega) < \alpha^+_{\lim} \dot{W}_{ext} \right) \tag{4.103}$$

In eq.(5.30) external power is deterministic and we denote by $\alpha^+_{\lim} \dot{W}_{ext}$.

$$P_f^{total} = 1 - \psi^{total} = \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{N_e} \sqrt{\frac{2}{3}} r_i(\omega) \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} < \alpha^+_{\lim} \sum_{k=1}^{m} \sum_{i=1}^{N_e} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} \right) \tag{4.104}$$

Following the last constraint of (3.49) we have

$$\begin{aligned} P_f^{total} = 1 - \psi^{total} &= \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{N_e} \sqrt{\frac{2}{3}} r_i(\omega) \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} < \alpha^+_{\lim} \sum_{k=1}^{m} \sum_{i=1}^{N_e} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} \right) \\ &= \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{N_e} \sqrt{\frac{2}{3}} r_i(\omega) \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} < \alpha^+_{\lim} \right) \\ &= 1 - \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{N_e} \sqrt{\frac{2}{3}} r_i(\omega) \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \geq \alpha^+_{\lim} \right) \end{aligned} \tag{4.105}$$

From the first constraint of (3.49) we have

$$P_f^{total} = 1 - \psi^{total} = 1 - \psi_r \tag{4.106}$$

Equation (5.33) shows that the total reliability ψ^{total} of the structure is equal to the chosen reliability ψ_r.

- **Load is random and strength is deterministic**

The stochastic problem (3.74) is formulated in section (3.2.2):

$$\alpha^{+} = \min \frac{\dot{W}_{in}}{\dot{W}_{ext}(\omega)}$$

$$\text{with} \quad \dot{W}_{in} = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}$$

$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall k = \overline{1, m} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} & \forall i = \overline{1, Ne}, \quad \forall k = \overline{1, m} \\ \text{Prob}\left(\dot{W}_{ext}(\omega) = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^{T} \mathbf{t}_{ik}(\omega) < 1 \right) = \psi_p \end{cases}$$

It is clear that the failure probability of the structure is the probability that the external power is larger than unit, which means:

$$P_f^{total} = 1 - \psi^{total} = \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^{T} \mathbf{t}_{ik} \geq 1 \right) = 1 - \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^{T} \mathbf{t}_{ik} < 1 \right) = 1 - \psi_p \tag{4.107}$$

In this case we have

$$\psi^{total} = \psi_p \tag{4.108}$$

Equation (5.35) shows that the total reliability of the structure is equal to the chosen reliability ψ_p.

- **When both strength and load are random**

Failure occurs for $\dot{W}_{in}(\omega) / \dot{W}_{ext}(\omega) < \alpha_{\lim}^{+}$ which is equivalent to $\dot{W}_{in}(\omega) - \alpha_{\lim}^{+} \dot{W}_{ext}(\omega) < 0$. The total failure probability of the structure is

$$P_f^{total} = \text{Prob}(M < 0) \tag{4.109}$$

where

$$M = \dot{W}_{in}(\omega) - \alpha_{\lim}^{+} \dot{W}_{ext}(\omega) \tag{4.110}$$

Because $\dot{W}_{in}(\omega), \dot{W}_{ext}(\omega)$ follows a normal distribution it is clear that from (5.37) M follows normal distribution. We have:

$$\begin{aligned} P_f^{total} &= \text{Prob}(M < 0) \\ &= \text{Prob}\left(\frac{M - \mu_M}{\sigma_M} < \frac{0 - \mu_M}{\sigma_M} \right) \\ &= \Phi\left(-\frac{\mu_M}{\sigma_M} \right) \end{aligned} \tag{4.111}$$

where μ_M, σ_M are the mean value and the standard deviation of M.

Calculation of the mean value μ_M *:*

$$\mu_M = \mu_{\dot{W}_{in}(\omega)} - \alpha^+_{\lim}\mu_{\dot{W}_{ext}(\omega)} \tag{4.112}$$

in which $\mu_{\dot{W}_{in}(\omega)}, \mu_{\dot{W}_{ext}(\omega)}$ are the mean value of $,\dot{W}_{in}(\omega), \dot{W}_{ext}(\omega)$ respectively. They are computed as in (3.56) and (3.79):

$$\mu_{\dot{W}_{in}(\omega)} = \sum_{k=1}^{m}\sum_{i=1}^{N_e}\sqrt{\frac{2}{3}}r_i\sqrt{\dot{\mathbf{e}}^T_{ik}\dot{\mathbf{e}}_{ik}} \qquad \mu_{\dot{W}_{ext}(\omega)} = \sum_{i=1}^{Ne}\sum_{k=1}^{m}\left(\dot{\mathbf{e}}^T_{ik}\boldsymbol{\mu}_{ik}\right)$$

Therefore

$$\mu_M = \mu_{\dot{W}_{in}(\omega)} - \alpha^+_{\lim}\mu_{\dot{W}_{ext}(\omega)} = \sum_{k=1}^{m}\sum_{i=1}^{Ne}\left[\sqrt{\frac{2}{3}}r_i\sqrt{\dot{\mathbf{e}}^T_{ik}\dot{\mathbf{e}}_{ik}} - \alpha^+_{\lim}\left(\dot{\mathbf{e}}^T_{ik}\boldsymbol{\mu}_{ik}\right)\right] \tag{4.113}$$

Calculation of the standard deviation of M :

Because M is linear combination of two random variables $\dot{W}_{in}(\omega), \dot{W}_{ext}(\omega)$. We can compute variance of M :

$$\mathrm{Var}(M) = \mathrm{Var}(\dot{W}_{in}) + \left(\alpha^+_{\lim}\right)^2\mathrm{Var}(\dot{W}_{ext}) \tag{4.114}$$

and the standard deviation of M :

$$\sigma_M = \sqrt{\mathrm{Var}(M)} = \sqrt{\mathrm{Var}\left[\dot{W}_{in}(\omega)\right] + \left(\alpha^+_{\lim}\right)^2\mathrm{Var}\left[\dot{W}_{ext}(\omega)\right]}$$

$$\sigma_M = \sqrt{\left[\sigma_{\dot{W}_{in}(\omega)}\right]^2 + \left(\alpha^+_{\lim}\right)^2\left[\sigma_{\dot{W}_{ext}(\omega)}\right]^2}$$

(4.115)

The standart deviations of $\sigma_{\dot{W}_{in}(\omega)}, \sigma_{\dot{W}_{ext}(\omega)}$ are computed as in (3.58) and (3.80):

$$\sigma_{\dot{W}_{in}(\omega)} = \sum_{k=1}^{m}\sum_{i=1}^{Ne}\sqrt{\frac{2}{3}}\sigma_r\sqrt{\dot{\mathbf{e}}^T_{ik}\dot{\mathbf{e}}_{ik}}$$

$$\sigma_{\dot{W}_{ext}(\omega)} = \sum_{i=1}^{Ne}\sum_{k=1}^{m}\sqrt{\dot{\mathbf{e}}^T_{ik}\mathbf{V}_{ik}\dot{\mathbf{e}}_{ik}}$$

So we have :

$$\sigma_M = \sqrt{\sum_{k=1}^{m}\sum_{i=1}^{Ne}\left[\frac{2}{3}\sigma_r^2\dot{\mathbf{e}}^T_{ik}\dot{\mathbf{e}}_{ik} + \left(\alpha^+_{\lim}\right)^2\dot{\mathbf{e}}^T_{ik}\mathbf{V}_{ik}\dot{\mathbf{e}}_{ik}\right]} \tag{4.116}$$

Look at the equivalent deterministic problem (3.82):

$$\min \frac{\dot{W}_{in}}{\dot{W}_{ex}} \left(= \alpha^{+}\right)$$

$$\text{with} \qquad \dot{W}_{in} = \sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} \left(r_i - \kappa_r \sigma_i \right) \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik} + \varepsilon_0^2}$$

$$\text{s.t.:} \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall i = \overline{1, N_e} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} = \mathbf{0} & \forall i = \overline{1, N_e}, \quad \forall k = \overline{1, m} \\ \dot{W}_{ex} = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^{T} \boldsymbol{\mu}_{ik} + \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^{T} \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) = 1 \end{cases}$$

from which we have

$$\dot{W}_{in} = \alpha_{\lim}^{+} \dot{W}_{ext} \tag{4.117}$$

$$\to \sum_{k=1}^{m} \sum_{i=1}^{Ne} \left(\sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik}} - \sqrt{\frac{2}{3}} \kappa_r \sigma_r \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik}} \right) = \alpha_{\lim}^{+} \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\dot{\mathbf{e}}_{ik}^{T} \boldsymbol{\mu}_{ik} + \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^{T} \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) \tag{4.118}$$

$$\to \sum_{k=1}^{m} \sum_{i=1}^{Ne} \left(\sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik}} - \alpha_{\lim}^{+} \dot{\mathbf{e}}_{ik}^{T} \boldsymbol{\mu}_{ik} \right) = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\sqrt{\frac{2}{3}} \kappa_r \sigma_r \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik}} + \alpha_{\lim}^{+} \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^{T} \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) \tag{4.119}$$

So the mean of M :

$$\mu_M = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\sqrt{\frac{2}{3}} \kappa_r \sigma_r \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik}} + \alpha_{\lim}^{+} \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^{T} \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right) \tag{4.120}$$

The substitution of (5.43), (5.47) in (5.38), the failure probability of entire structure can be computed as

$$P_f^{total} = \Phi\left(-\frac{\mu_M}{\sigma_M} \right) = \Phi\left[-\frac{\sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\sqrt{\frac{2}{3}} \kappa_r \sigma_r \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik}} + \alpha_{\lim}^{+} \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^{T} \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right)}{\sqrt{\sum_{k=1}^{m} \sum_{i=1}^{Ne} \left[\frac{2}{3} \sigma_r^2 \dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik} + \left(\alpha_{\lim}^{+} \right)^2 \dot{\mathbf{e}}_{ik}^{T} \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} \right]}} \right]$$

(4.121)

The reliability of the entire structure relates with the parital reliabilities $\psi_r = \Phi^{-1}(\kappa_r)$, $\psi_p = \Phi^{-1}(\kappa_p)$ through following equation:

$$\psi^{total} = 1 - P_f^{total} = \Phi\left(\frac{\mu_M}{\sigma_M} \right) = \Phi\left[\frac{\sum_{i=1}^{Ne} \sum_{k=1}^{m} \left(\sqrt{\frac{2}{3}} \kappa_r \sigma_r \sqrt{\dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik}} + \alpha_{\lim}^{+} \kappa_p \sqrt{\dot{\mathbf{e}}_{ik}^{T} \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik}} \right)}{\sqrt{\sum_{k=1}^{m} \sum_{i=1}^{Ne} \left[\frac{2}{3} \sigma_r^2 \dot{\mathbf{e}}_{ik}^{T} \dot{\mathbf{e}}_{ik} + \left(\alpha_{\lim}^{+} \right)^2 \dot{\mathbf{e}}_{ik}^{T} \mathbf{V}_{ik} \dot{\mathbf{e}}_{ik} \right]}} \right]$$

(4.122)

Chapter 6

Conclusions and further studies

6.1 Conclusions

The thesis contributes an approach to direct probabilistic structural limit state design with the limit states plastic collapse, ratcheting, and alternating plasticity. These limit states can be obtained by limit analysis and in the case of time variant loads by shakedown analysis as solutions of a mathematical program (optimization), which is linear for Tresca yield function and nonlinear for von Mises yield fucntion. With respect to the uncertainties, limit and shakedown analysis have the great advantage that they make the problem time-invariant and show that many uncertain quantities have no influence on the above limit states.

Different to reliability analysis which calculates the failure probability of a given design, a chance constrainted programming (CCOPT) approach has been developed to make the design decision on the basis of the reqiured reliability (or failure probability). In the general case chance constrained programming is a hard problem because probabilities have to be calculated as high dimensional integrals during the optimization algorithm.

Due to the linearity of chance constraints on random variables and their Gaussian distribution, the deterministic equivalents can be obtained for both linear and nonlinear programming. A deterministic equivalent could also be found for lognormally distributed variable strength through duality of upper bound and lower bound shakedown loads.

With chance constrained programming techniques, the stochastic shakedown problem is transformed into an equivalent deterministic problem and then solution can be obtained by NLP framework. The thesis developed successfully three algorithms to treat large-scale shakedown analysis problems with random strength and load variables:

- The dual algorithm (A1) is employed to compute simultaneously the upper bound and lower bound shakedown loads in the situations:
    - Both strength and load are deterministic.
    - The load acting on the structure is deterministic, the strength of structure is distributed normally or lognormally.
- The kinematic algorithm (A2) can be used to solve shakedown problems in following situations:
    - Both strength and load are deterministic.
    - The load acting on the structure is deterministic, only the strength is distributed normally or lognormally.
    - The strength is deterministic, the load is normally distributed.
    - The strength and the load are both random with normal distribution.

  - The strength is random with a lognormal distribution and the load is random with a normal distribution.
- The dual algorithm (A3) is employed to compute simultaneously the upper bound and lower bound shakedown loads of a Kirchhoff-Love plate subjected to bending in the situations:
  - Both strength and load are deterministic.
  - The load acting on plate is deterministic, the strength of plate is distributed normally or lognormally.

In these algorithms, a combination of penalty methods and Lagrange multipliers methods are used to handle the incompressibility, compatibility and normalization conditions. Newton's method is employed to solve the resulting system of nonlinear equations extending the Karush-Kuhn-Tucker conditions.

For engineering design, structural reliability is a post design problem while stochastic programming is a pre-design problem. In the simple case of only one uncertain strength variable or only one random load variable, reliability analysis is "invers" to chance constrained programming and can be used to check the latter as seen in chapter 5. In the same way numerical reliability analysis can be used to check any constrained programming solution for normally or lognormally distributed variables. It is found that the load factor decreases quickly with increasing coefficient of variation of the strength and load.

One result is that the load factors for normally distributed strength is always larger than for lognormally distributed strength. Therefore, working with the simpler normal distribution will give safe results which is most important for engineering applications. This makes the method more transparent to many engineers and it is easierly extended to the case that the strengths in different points of the structure are correlated (stochastic field).

The proposed chance constained programming approach to limit design has a great practical advantage over the structural reliability analysis which is performed after the structural design. It is well known that the relibalibility is strongly influenced by the tails of the distributions and the failure probability P_f can change over magnitudes for different stochastic models and small change of stochastic data. It has been pointed out that the reliability index β is much less sensitive and the sensitivity is a consequence of the function $P_f = \Phi(-\beta)$. This good behaviour of β carries over to κ and as a consequence to the engineering decision of the designed limit or shakedown load factor.

6.2 Future studies

This work investigates shakedown problems with uncertainty of loading by the kinematic theorem as shown in chapter 3. Due to the linearity of chance constraints on random variables and their Gaussian distribution (the random varialbes are stress vectors) the deterministic equivalents can be obtained by a direct computation method of CCOPT. The non-Gaussian distribution of loading needs futher studies. For non-Gaussian loading it is an open problem to find a deterministic equivalent of the probabilistic constraint in the lower bound approach

$$\mathrm{Prob}\left[\, f(\boldsymbol{\sigma}) < \sigma_0 \,\right] = \mathrm{Prob}\left[\sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2} < \sigma_0\right] = \psi \ .$$

Therefore, the further study on this issue needs still to be done and maybe analytical approximation of CCOPT or numerical calculation of the probability has to be used here.

This approach applies generally to 2D, 3D structures. The upper bound method may be implemented with any displacement-based finite elements. In this thesis, plane stress problems are used to demonstrate the conceptual power and numerical efficiency of the method. Edge-based SFEM T3 elements were applied to discretize 2D structures, for 3D structure, face-based SFEM may be applied and this could be a future work. Stress-based finite element are more appropriate for lower bound methods. They are much less developed and only recently proposed for SFEM.

For plate structures, in this thesis, a dual algorithm was developed to calculate upper bound and lower bound shakedown loads in case of random strength, while the loads are still deterministic. Further study may develop a kinematic algorithm to treat stochastic shakedown problems of thin plates under random strength and loads. This approach may be extended to thick plate structures (*Mindlin-Reissner* theory of *plates)*

The newly proposed equilibrium cell-based SFEM could be used to derive true lower bound methods. Other numerical methods may be used to formulate problems such as the meshless element-free Galerkin (EFG) method or the iso-geometric finite element method (IGA).

This approach may be used for another type of structures like trusses and frames (1D elements) or shells (2D elements).

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